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EFFECT OF TEST MACHINE EXTENSIONAL RIGIDITY ON THE INITIAL BUCKLING LOAD OF UNREINFORCED CIRCULAR CYLINDRICAL SHELLS IN AXIAL COMPRESSION

By

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November 1971

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U. S. ARMY AIR MOBILITY RESEARCH AND DEVELOPMENT LABORATORY
FORT EUSTIS, VIRGINIA

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The inadequacy of small displacement theory of thin shells in explaining the buckling of circular cylindrical shells has been established. Likewise, it has become general belief that an explanation can be given by means of a nonlinear large deflection theory. The current theories lean heavily upon a plausible but arbitrarily chosen energy criterion first given by Tsien, and predict that the initial buckling load should be influenced by the testing machine stiffness. Two experiments described, statistically analyzed, and discussed by the authors in a previous paper revealed that test machine extensional rigidity does not influence the initial failing load. These results were obtained from repeated tests on a single near-perfect aluminum specimen and individual tests on many less perfect steel shells. They had R/t ratios of 313 and 226, respectively, and buckled at 77 and 45 percent of the classical critical load.

Two additional experiments are described in detail in this report supplement. They extend the authors' previous research. Circular cylinders with higher R/t ratios which buckled at a lower percentage of the classical critical load were used. The shells had R/t ratios of 946 and 1419 and buckled at 43.9 and 24.6 percent of classical, respectively. In all four experiments, 204 tests were conducted. Ranges in test machine extensional stiffness from 589,000 lb/in. to 2400 lb/in. were considered; R/t values ranged from 226-1419 and P_{cr}/P_{cl} values from .249 to .77. L/D ratios were approximately constant in all cases. The results show that the Tsien criterion is inapplicable. This may be due to the invalidity of the criterion itself or to the inadequacy of the large displacement analysis. Thus, scatter in experimental results is not due to the influence of test machine rigidity, and a new look at the large displacement analysis and the appropriate criterion is needed.

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	ROLE	WT	ROLE	WT	ROLE	WT
Initial Buckling Load Testing Machine Extensional Stiffness Ratio of Radius to Wall-Thickened, R/t Ratio of Critical Buckling Load/Classical Buckling Load, P_{cr}/P_{cl}. Ratio of Length of Specimen to its diameter, L/D Regression Analysis Level of Significance Test Statistic Operating Characteristic Curve (OC Curve)						

DEPARTMENT OF THE ARMY
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The report has been reviewed by this Directorate and is considered to be technically sound. It is published for the exchange of information and the stimulation of future research.

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ABSTRACT

The inadequacy of the classical small displacement theory of thin shells in explaining the buckling phenomenon for circular cylindrical shells under axial compression and spherical shells under external pressure loading has long been established. Likewise, it has become general belief that an explanation of their behavior can be given by means of a nonlinear large deflection theory. The current interpretations of such theories when applied to these problems lean heavily upon a plausible but arbitrarily chosen energy criterion first given by Tsien, and predict that the initial buckling load of thin shells should be influenced by the testing machine stiffness. Two experiments described, statistically analyzed, and discussed by the authors in a previous report revealed that test machine extensional rigidity does not influence the initial failing load of axially compressed cylinders to a high degree of probability. These results were obtained from repeated tests on a single near-perfect aluminum specimen and individual tests on many less-perfect steel shells. They had R/t ratios of 313 and 226, respectively, and buckled at 77 and 45 percent of the classical critical load.

Two additional experiments using the single specimen approach described in detail in this report supplement and extend the authors' previous research by considering circular cylinders with higher R/t ratios which buckled at a lower percentage of the classical critical load. The shells had R/t ratios of 946 and 1419 and buckled at 43.9 and 24.6 percent of classical, respectively. In all 4 experiments, 204 tests were conducted. Ranges in test machine extensional stiffness from 589,000 lb/in. to 2400 lb / in. were considered; R/t values ranged from 226 to 1419, and P_{cr}/P_{cl} values ranged from .249 to .77. L/D ratios were approximately constant in all cases. The results provide overwhelming evidence that the Tsien criterion is inapplicable in all problems considered. This may be due to the invalidity of the criterion itself or to the inadequacy of the large displacement analysis. The consequences, however, are the same; scatter in experimental results is not due to the influence of test machine rigidity, and a new look at the large displacement analysis and the appropriate criterion is needed.

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LIST OF SYMBOLS

k	number of values within each set
n	number of sets
P_{cr}	buckling load for the shell
P_{cl}	classical buckling load for the shell
R/t	radius-to-thickness ratio for a shell
S^2	sample variance
$\bar{x}$, $\bar{y}$	arithmetic means
α	level of significance
β	a parameter (probability of occurrence of type II error)
σ	standard deviation

INTRODUCTION

The veracity of the commonly accepted Tsien criterion[1] for the instability of shell bodies has been examined by the authors in recent publications.[2,3,4] According to this belief--constant total potential energy before and after buckling--there should be a difference in the initial axial buckling load of a circular cylindrical shell between tests made in rigid and dead-weight testing machines. These two extreme cases are illustrated by the dotted lines in the equilibrium curve of the Karman-Tsien[5] postbuckling theory shown in Figure 1. Normal elastic testing machines are typified by the solid line. It was pointed out by these authors that the elastic characteristic of the testing machine might be a cause of the large scatter of the data obtained by different experimenters.

PRIOR RESEARCH ON THE PROBLEM

Prior to the research cited above, little experimental work had been carried out to check this premise. The first experiments with a direct bearing on the issue appear to be those of Horton, Johnson, and Hoff.[6] These tests showed that the effect was questionable and indicated strongly the need for a more intensive program. A later study by Mossakovskii and Smelyi[7] appeared to verify the Karman-Tsien theory, but on close examination the finality of the conclusion is marred by the paucity of data on which it was founded. Almroth, Holmes and Brush[8] noted that the test vehicle characteristics appeared to be more important than those of the test machine; but this sound comment can only be considered as qualitative, since no quantitive data were presented. Likewise, the observation of Krenzke[9] that his repeated tests on a single plastic sphere gave no evidence to support Tsien's criterion can only be regarded as a justification for further study.

EXPERIMENTAL APPROACH FOR FURTHER RESEARCH

The extensive experimental studies reported in Reference 2 have revealed, after a careful statistical reduction of the data, that test machine extensional rigidity does not influence the initial failing load of axially compressed cylinders to a high degree of probability. These results were obtained from repeated tests on a single near-perfect aluminum specimen (test series B) and individual tests on many less-perfect steel shells (test series A). They had R/t ratios of 313 and 226, respectively, and buckled at 77 and 45 percent of the classical critical load.

It could be argued that in these two tests the ranges of R/t and P_{cr}/P_{cl} considered exclude those of the greatest practical interest. Likewise, it might be asserted that the audio-visual method of buckle determination used there was inadequate. To counter these possible criticisms, two additional experiments (test series C and D) were conducted using the single specimen approach; they are described in detail in this report. The shell specimens had R/t ratios of 946 and 1419 and buckled at 43.9 and 24.6 percent of classical, respectively.

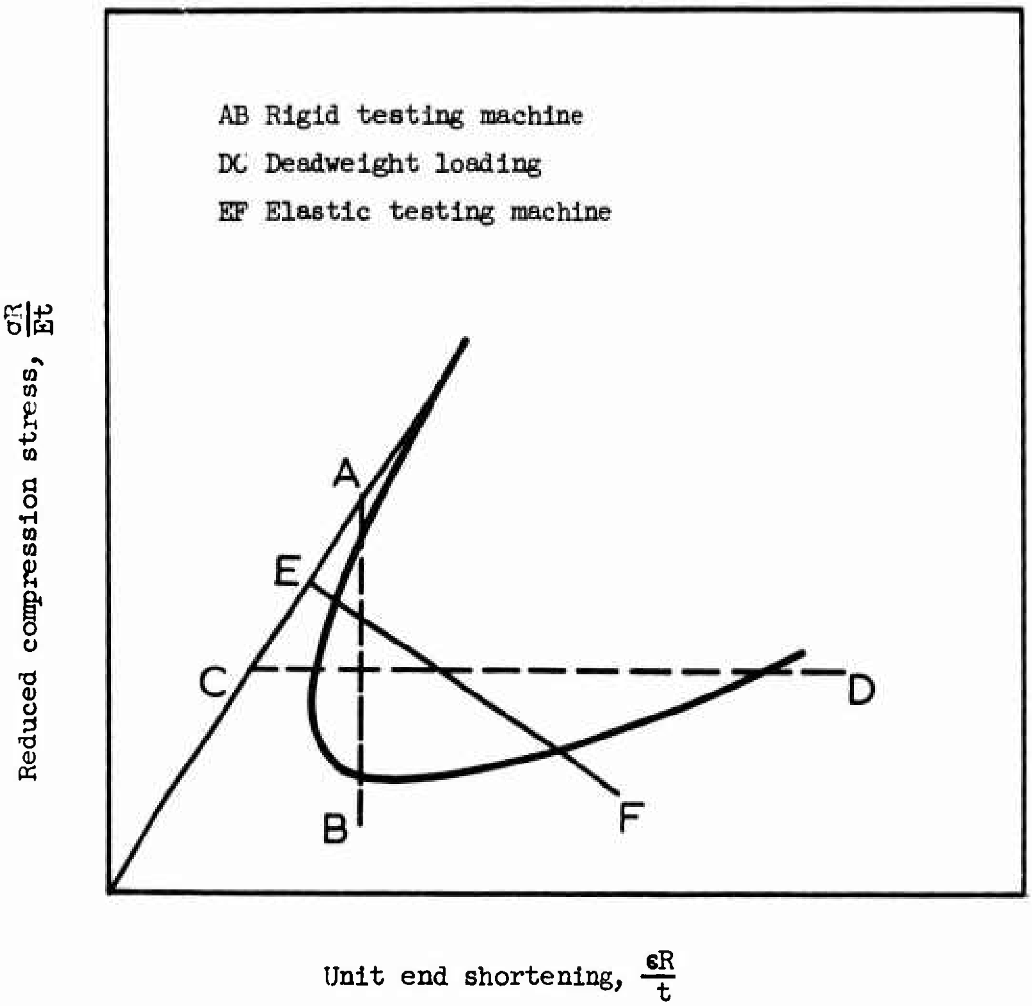


Figure 1. Equilibrium Curve for Buckling of a Thin Circular Cylindrical Shell Unver Uniform Axial Compression.

DETAIL OF THE TEST SPECIMENS

Both specimens were made from 12-inch-wide precision-rolled shim steel material. One test vehicle was made from 2-mil stock and the other from 3-mil stock. The actual shells were manufactured by a wraparound and seam technique, using an aluminum mandrel which had been accurately machined between centers to a diameter of 5.677 inches with less than 3/10,000-inch taper in the full 12-inch length. The flat sheets, cut to proper size, were lap-jointed and soft soldered. Removal from the manufacturing mandrel was readily accomplished by slightly tapping or cooling the mandrel-specimen assembly. The setup is shown in Figure 2.

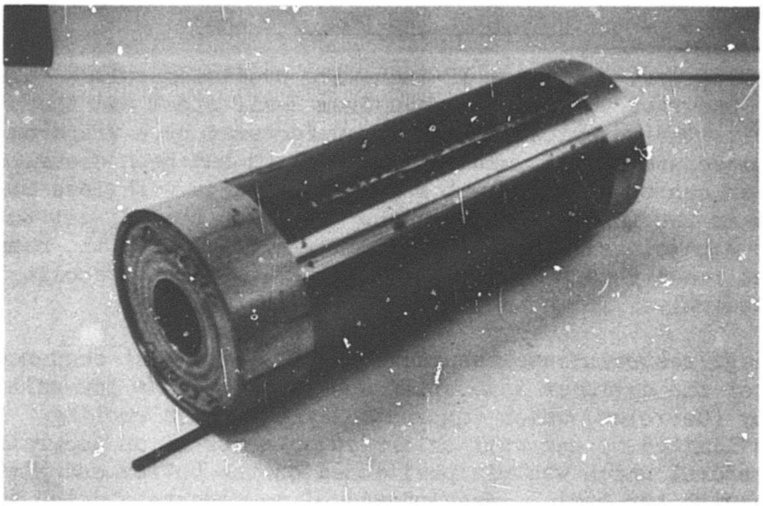
Constant edge restraint was assured throughout the test sequence by casting the edges of the cylinder into stiff end plates with a low-melting temperature alloy (Cerrolow) which contracts slightly upon cooling. Inward buckle motion was limited to one wall thickness by means of an accurately machined interior mandrel which was pin positioned on the bottom end plate to remain concentric with the shell. The individual components and the assembled unit are shown in Figure 3.

TEST ENVIRONMENT

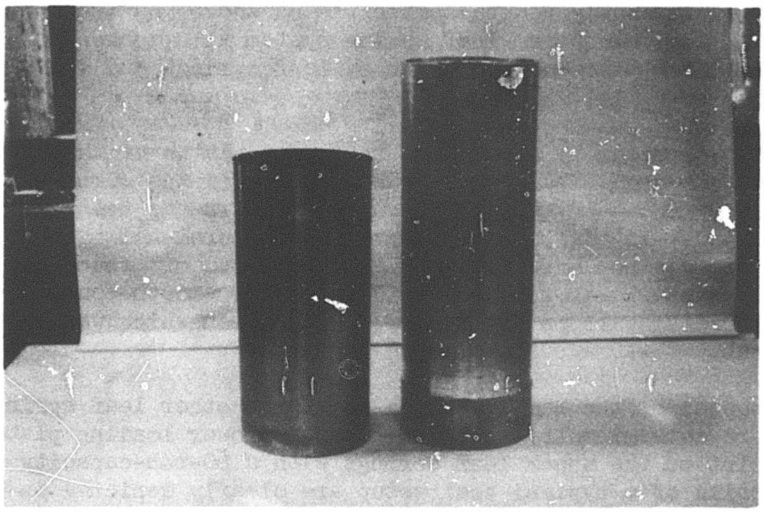
The Baldwin-Lima-Hamilton Universal testing machine used in test series A was again employed here. Its extensional stiffness variations were obtained using the previously adopted principle as shown in Figure 4(a). It consisted of a simple steel leaf spring system with a supporting and clamping bracket which allowed the number of leaf springs and their support positions to be varied.

The bracket was securely mounted to the upper platen of the testing machine, as shown in Figure 4(b). This permitted the number of leaf springs to be varied without any alteration whatsoever in the test setup of the specimen below. In addition, the lateral and torsional rigidity of the basic machine was not affected. A magnetized pad of ground tool steel positioned on the lower spring provided a hard, smooth contact point for the steel loading ball on the top of the specimen which was identical from test to test.

The stiffnesses of the basic machine and seven other leaf spring modifications were determined by forcing apart the lower loading platen and contact point on the upper leaf springs with a 60-ton-capacity hydraulic jack. Details of a typical test setup are clearly depicted in Figure 5. Figure 5(c) shows the quarter-span supports which were used with the 6 leaf spring arrangement to provide increased rigidity. In Figure 5(d), a rigid steel block clamped between the single leaf spring and the supporting bracket developed the full rigidity of the basic machine. Care was taken to assure that the cross-beam of the testing machine was positioned at the height used in the testing sequence. The motion which resulted and the load which was induced were measured by two dial gages and the machine load scale, respectively. In all cases, the dial gages were placed symmetrically about the load point in the plane of the fixed cross-beam.

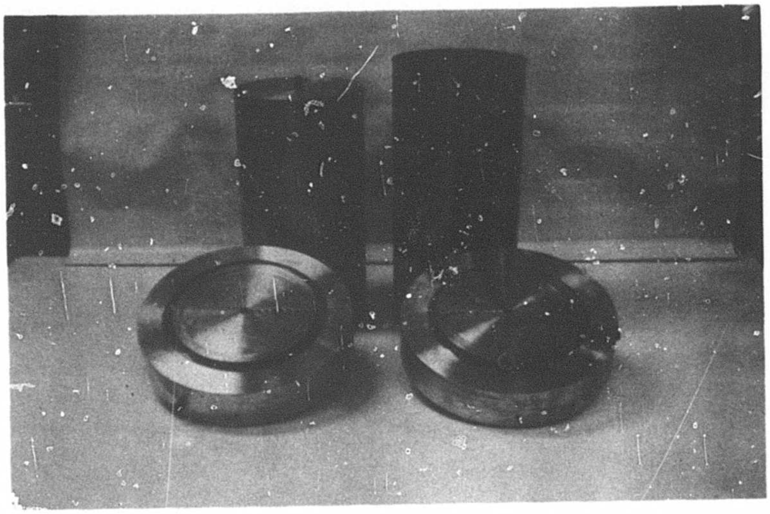


(a). View of Mandrel-Specimen Assembly
Showing Detail of Soft Solder Joint.

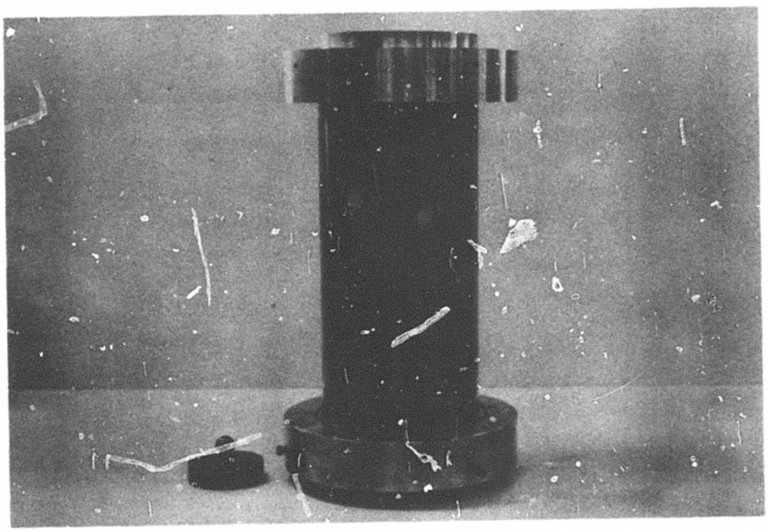


(b). Finished Specimen Removed From Manufacturing Mandrel.

Figure 2. Method of Making Shim-Steel Specimens - Test Series C and D.

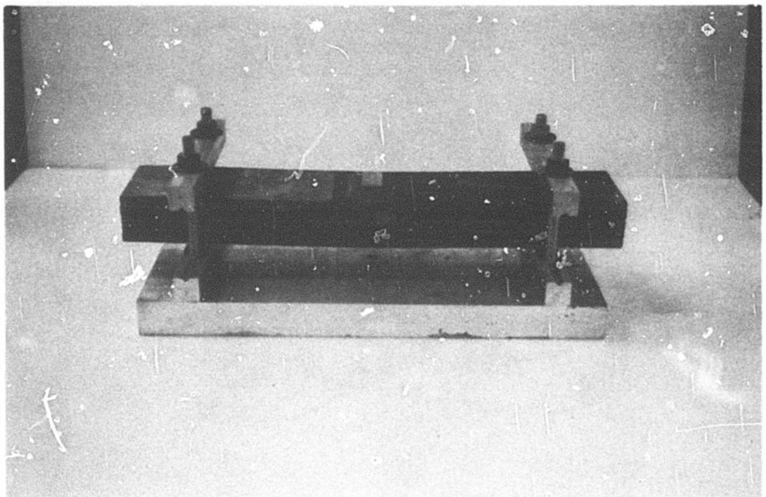


(a). View of Aluminum End Plates, Interior Mandrel and Specimen.

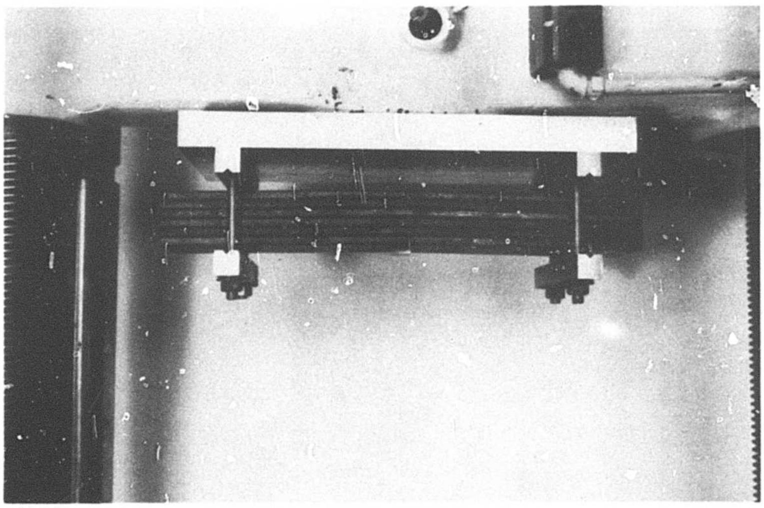


(b). Assembled Unit with Potted Ends Showing Steel Loading Ball.

Figure 3. Preparation of the Specimens for Testing - Test Series C and D.

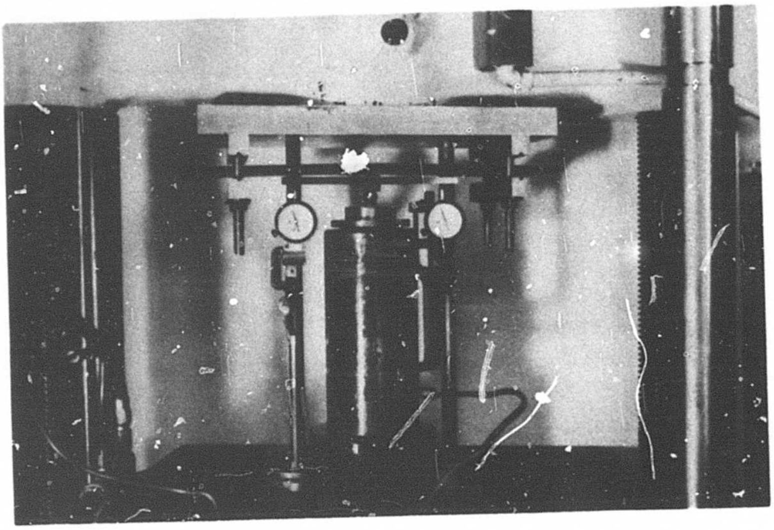


(a). View of Steel Leaf Spring System.

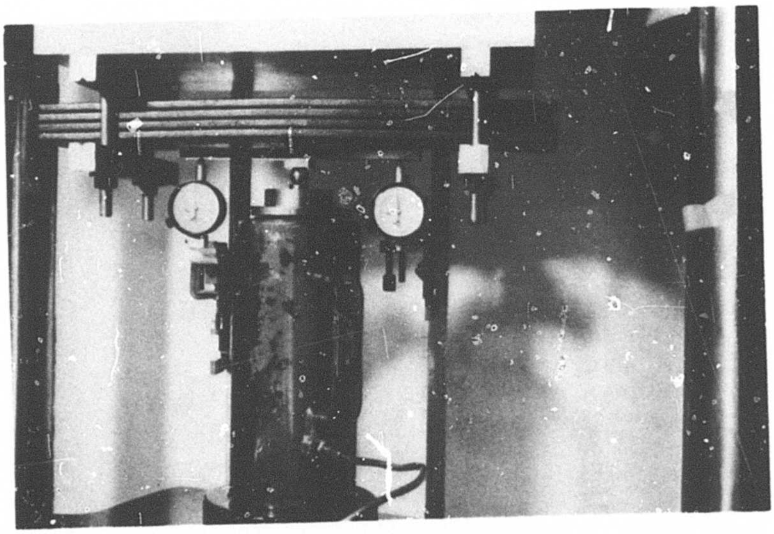


(b). Leaf Spring System Securely Mounted to the Upper Load Head of the Testing Machine.

Figure 4. Method of Modifying Extensional Stiffness of the 60,000-lb Baldwin-Lima-Hamilton Testing Machine - Test Series C and D.

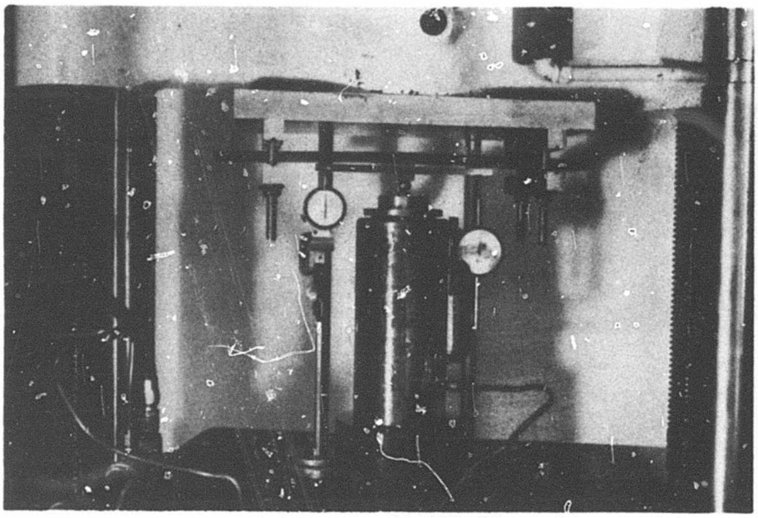


(a). Typical Test Setup.

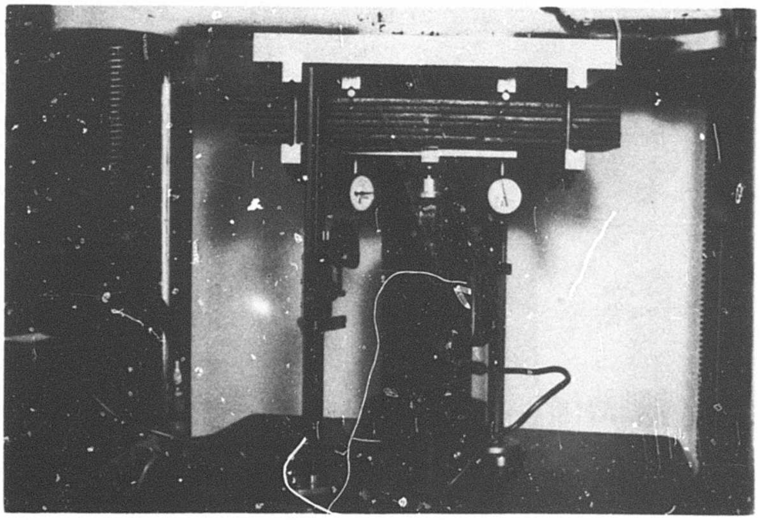


(b). Typical Arrangement of the Two Dial Gages Used to Transduce Deflections.

Figure 5. Method of Determining Composite Stiffness in Test Series C and D Using Dial Gages, Machine Load Cell and 60-Ton Hydraulic Jack.



(c). View of Quarter-Span Supports Used to Increase Rigidity.



(d). Steel Block in Position to Develop Full Stiffness of Basic Machine.

Figure 5. Continued.

The force-head separation data are recorded in Tables I through VIII. The composite stiffness of each configuration is determined from the load-average deflection plots shown in Figures 13 through 20 and is summarized in Table X. The table shows stiffness values ranging from 577,000 for 7,800 lb/inch--a 74:1 variation.

METHOD OF LOAD APPLICATION AND DETERMINATION OF INSTABILITY

Since all stiffness variations were accomplished by means of an overhead leaf spring system, no movement of the test vehicle was necessary during the entire test sequence. However, to assure uniformity of load distribution after each stiffness alteration, the end shortening of the cylinder was monitored at three equally spaced points around the periphery using sensitive strain gage deflectometers. The analog signals received from all three transducers were recorded simultaneously on a Sanborn recorder. A well-lubricated steel loading ball located on the top end plate was positioned in such a manner that changes in end shortening shown by these readings were equal to within 2/10,000 inch in the initial stages of loading.

Buckle determination was made by using an electro-optical noncontacting displacement probe to monitor displacement at a point on the shell wall. This device, a Fotonic Sensor, uses a fiber optic cable to direct a constant intensity light source on a moving surface, and it can detect variations in the amount of reflected light (Figure 6). Such equipment has good sensitivity and high resolution and can be used to measure displacements on the order of microns. In the test series it was positioned in the following manner. The shell was buckled in the base machine and shown to be invariant in load-carrying capability and buckle location. The probe was then positioned normal to the shell wall near the spot where initial buckling was known to occur. This procedure is shown in Figure 7. It was subsequently calibrated in place, and the output signal was monitored on the Sanborn recorder shown in Figure 8.

Figure 9 depicts the typical wall motion which resulted during loading of the 3-mil specimen. Load values were read from the Baldwin-Lima-Hamilton load cell and were instantly recorded on the strip chart by an electric impulse marker. The point of buckling is clearly defined. A similar load-radial deflection history is given in Figure 10 for the 2-mil specimens. This loading process was repeated six times at each of eight different levels of machine stiffness taken in a random fashion for both the 3-mil and the 2-mil test vehicles. Although the loading rate did not vary over a wide range during the test series, no effort was made to insure uniformity.

EXPERIMENTAL DATA

All critical load values obtained for the 3-mil specimen are given in Table X. Likewise, the buckling loads for the 2-mil cylinder are recorded in Table XII. Each test was conducted according to the procedure outlined in the preceding section.

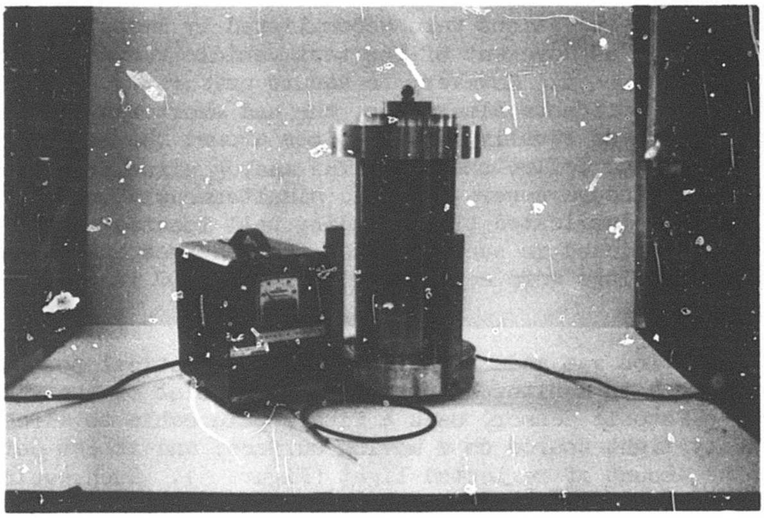


Figure 6. View of Fotonic Sensor Used to Transduce Radial Wall Motion and Establish Point of Instability - Test Series C and D.

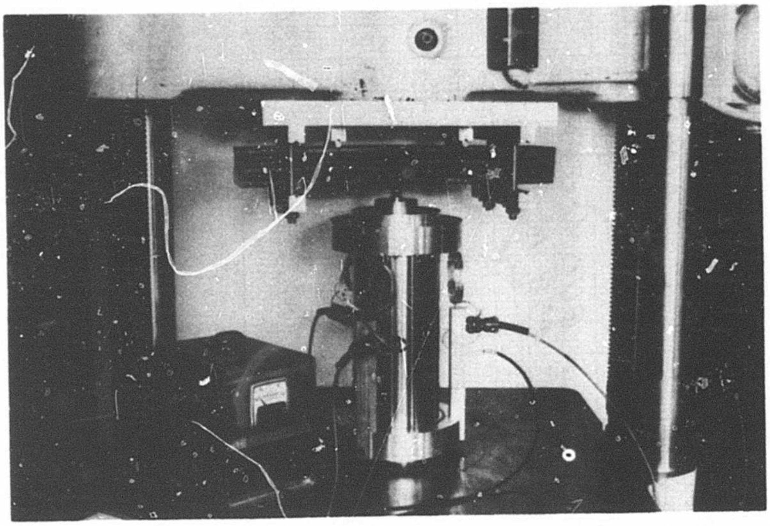


Figure 7. Close-Up View of a Typical Specimen and Associated Instrumentation - Test Series C and D.

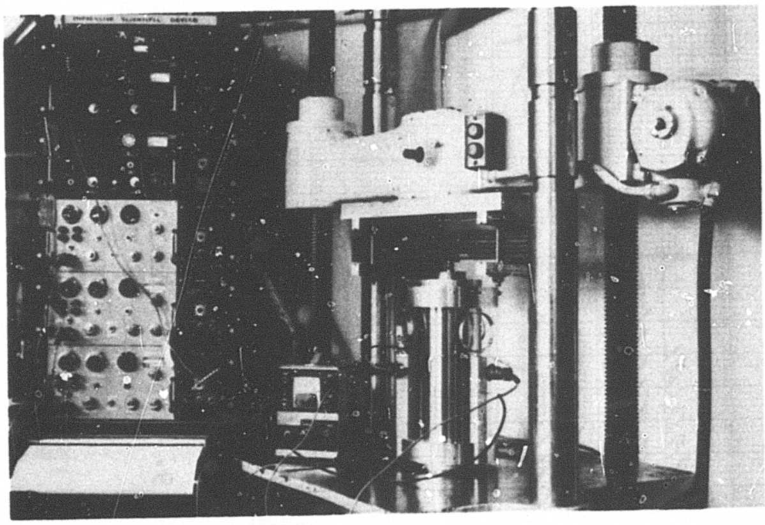


Figure 8. Typical Arrangement of Deflectometers and Sanborn Recorder - Test Series C and D.

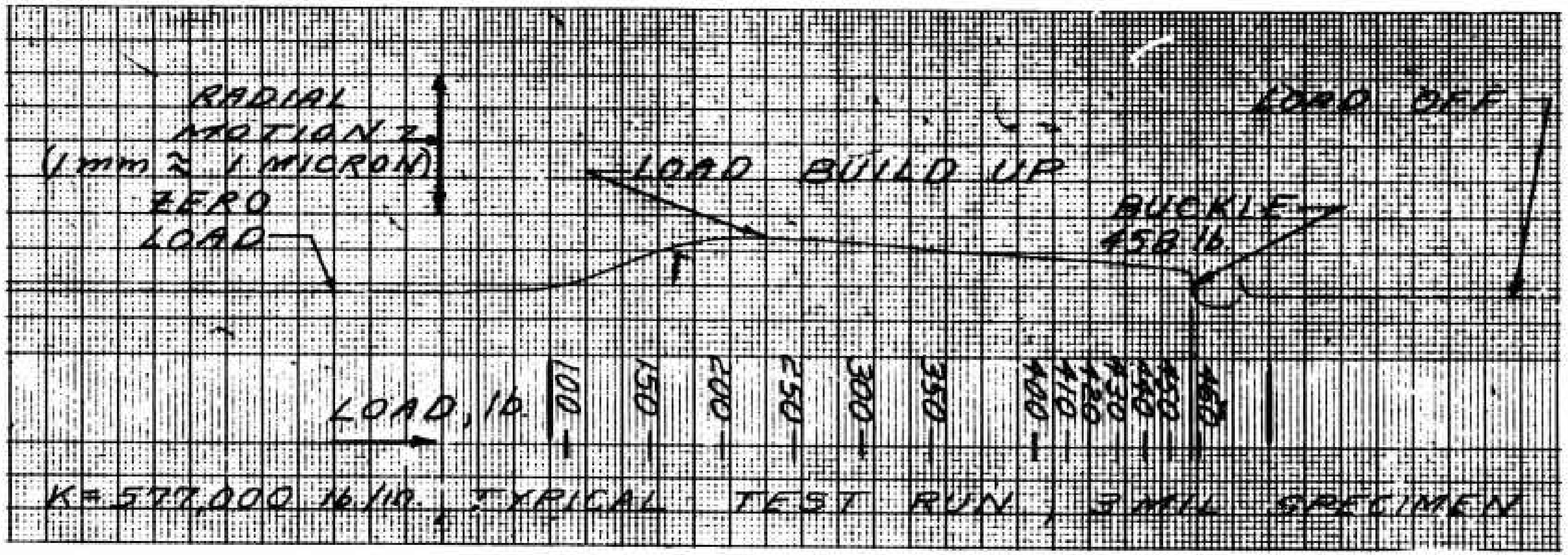


Figure 9. Typical Radial Motion of the Shell Wall During Axial Compression of the 3-Mil Specimen in Test Series C Transduced by the Fotonic Sensor and Recorded by a Sanborn Recorder.

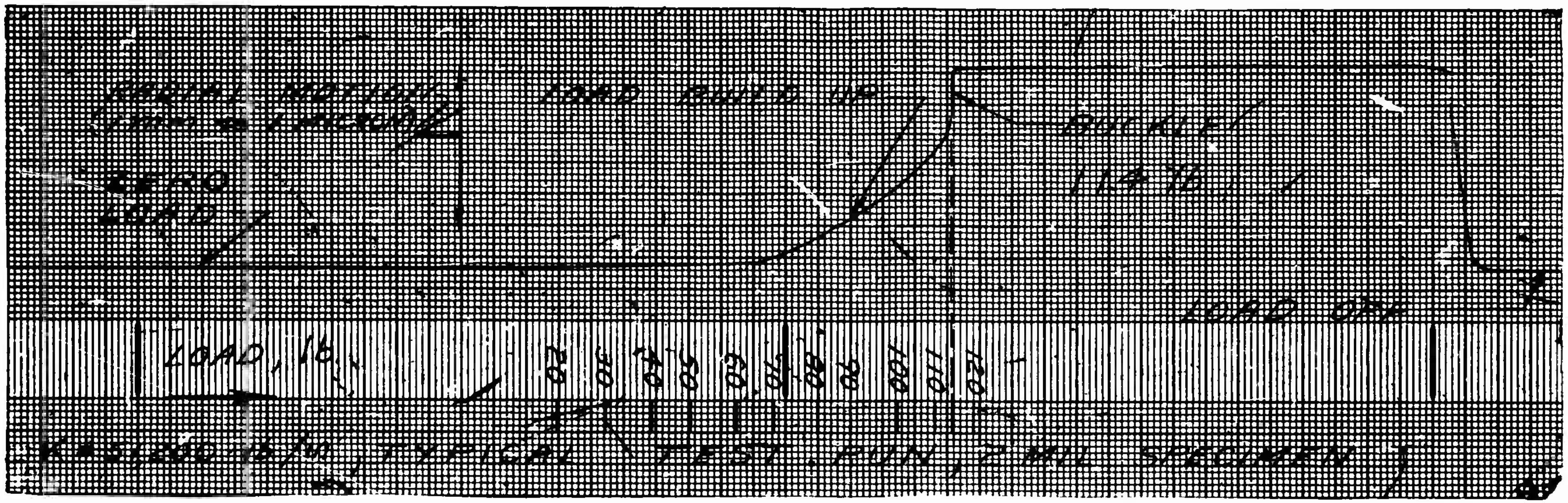


Figure 10. Typical Radial Motion of the Shell Wall During Axial Compression of the 2-Mil Specimen in Test Series D Transduced by the Fotonic Sensor and Recorded by a Sanborn Recorder.

The mean value of initial failing load from all 48 tests of the 3-mil test vehicle was 456.72 lb, giving a ratio of $(\bar{P}_{cr}/P_{cl}) = 0.439$. The corresponding value for the 2-mil shell was 112.59 lb, yielding a ratio of $(\bar{P}_{cr}/P_{cl}) = 0.246$.

DISCUSSION OF EXPERIMENTAL RESULTS

Since several levels of machine rigidity were considered in these last two test series, a regression analysis (adopted previously in test series B2) was again employed to examine the possibility of a relationship between the stability level and the machine extensional rigidity. Here, an even more powerful analysis can be made because of the greater number of degrees of freedom. The lengthy calculations are performed in Appendixes II and III, but the important results are presented below.

It is assumed that the buckling load may be expressed as a linear function of machine stiffness. The method of least squares leads to the empirical regression lines drawn in Figures 11 and 12 for the 3-mil and 2-mil buckling data, respectively. These lines are drawn through the discrete average buckling loads obtained for each set. The theoretical slopes β and the intercepts A' are examined statistically as before to check the initial assumption that buckling load is related to machine rigidity. The same significance tests used in test series B apply here, but a value of $k = 6$ must be used.

In addition, for these two cases it is desirable to compare the variation about the regression line with that existing within the sets of load values corresponding to the individual machine stiffness. This is done with the F test, and for both cases the hypothesis of linearity is accepted at the 95-percent confidence level. Additionally, the homogeneity of the several set variances is established by Cochran's test, and the test shows for both shells that the variances do not differ significantly. Thus, it is permissible to make a pool estimate of variance about the regression line and to establish a pooled standard deviation. The values of these quantities are 5.3939 lb.^{2} and 2.3225 lb. for the 3-mil shell and 4.0152 lb.^{2} and 2.004 lb. for the 2-mil shell. Thus, the standard deviation of slope and intercept can be computed. These values are 0.1862×10^{-5} in. and 0.394 lb for the 3-mil shell and 0.1606×10^{-5} in. and 0.340 lb. for the 2-mil shell. The 95-percent confidence interval for β is $[-0.5958; 0.1546] \times 10^{-5}$ in. and for the intercept is [455.56; 458.38] lb for the 3-mil specimen and $[-0.559, 0.088] \times 10^{-5}$ in. and [112.17; 113.53] lb, respectively, for the 2-mil shell. The t test statistics to check the hypotheses of zero slope and intercept equal to the average critical load are -1.185 and 0.623, respectively, for the 3-mil specimen and -1.468 and 0.772, respectively, for the 2-mil shell. The criterion for rejection of these hypotheses is

$$|t| \leq t_{\alpha/2;\ kn - 2}$$

where k is the number of tests within each set and n is the number of sets. At the 5-percent level of significance with $k = 6$ and $n = 8$,

$$t_{0.025;\ 46} = 2.015$$

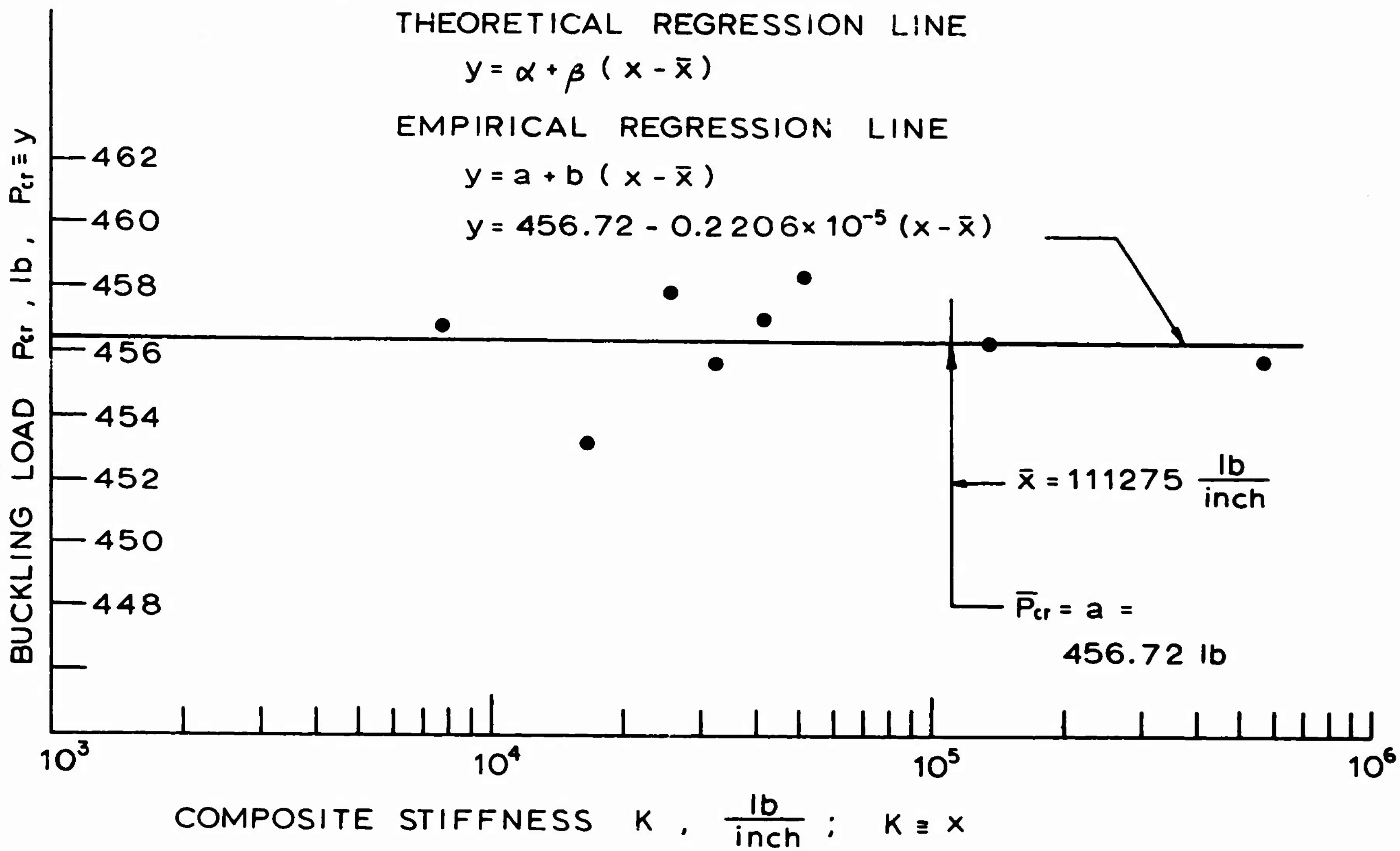


Figure 11. Regression Analysis for Test Series C; Buckling Load Considered as a Linear Function of Machine Stiffness.

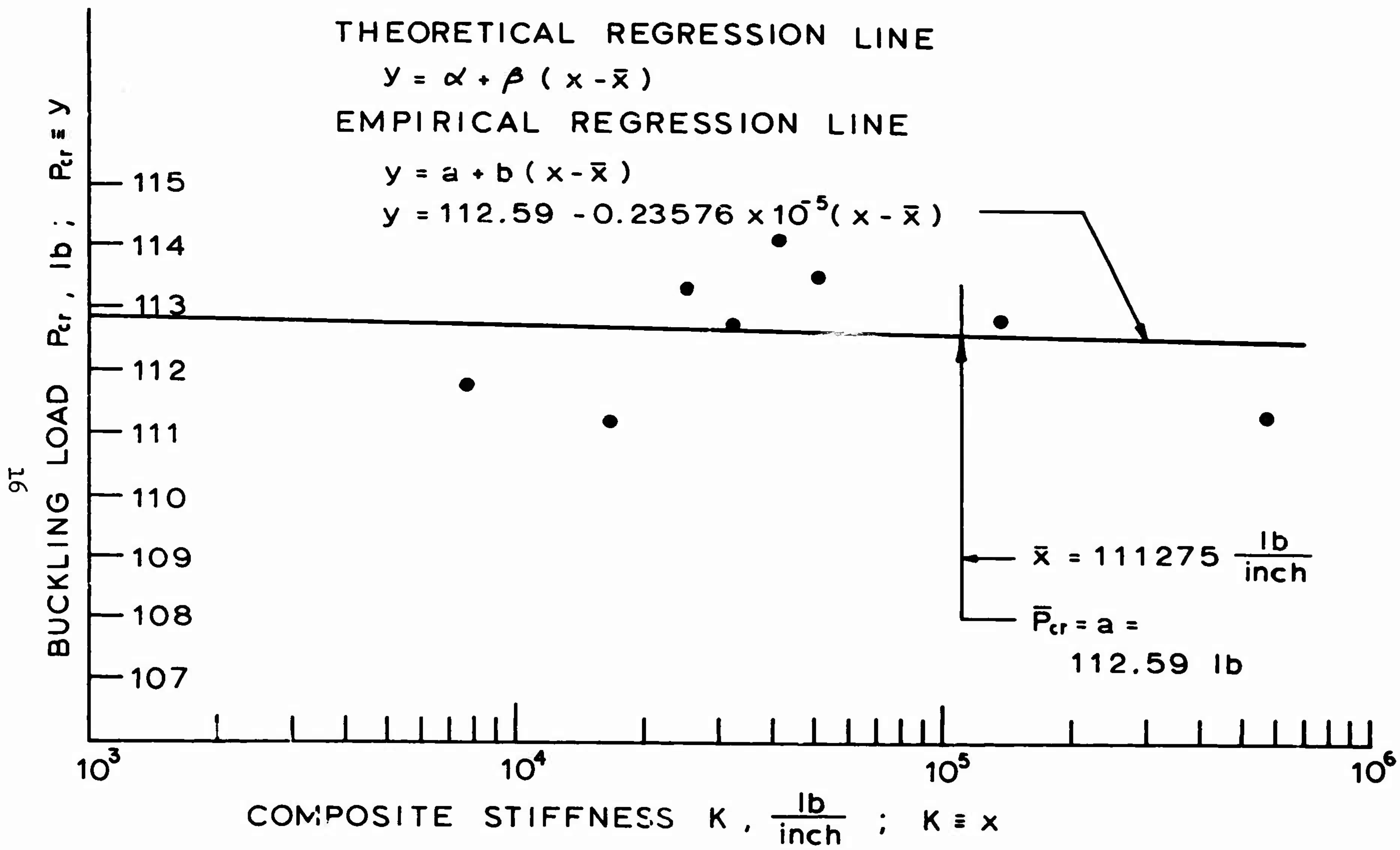


Figure 12. Regression Analysis for Test Series D; Buckling Load Considered as a Linear Function of Machine Stiffness.

Since

$$| \ 0.623 \ | < | \ 0.772 \ | < | \ -1.185 \ | < | \ -1.468 \ | < 2.015$$

all hypotheses are accepted at the 95-percent level.

The extreme sensitivity of the foregoing analysis is readily demonstrated. The operating characteristic curve for the t test at the 5-percent level of significance and a sample size of 48 shows a 95-percent probability that the hypotheses (the true slopes are zero) would be rejected if they differ from zero by as little as 0.7659×10^{-5} in. and 0.6606×10^{-5} in. for the 3-mil and 2-mil cylinders, respectively. These slopes imply changes in load value at zero stiffness of only 0.23 and 0.66 percent of the corresponding average critical load levels. Likewise, the same curve shows a 95-percent probability of rejecting the hypotheses that the true intercepts are equal to the mean values of buckling load when they differ from these values by as little as 0.35 and 1.24 percent of the corresponding averages.

CONCLUSIONS

The two experimental studies described in Reference 2 (test series A and B) and the two presented here (test series C and D) have produced results which are in absolute accord with each other. It is clear from the several statistical analyses that the data are of high quality, and the probability of error in the result is slight. The work demonstrates that test machine rigidity has no influence on initial instability load for circular cylindrical shells in the range of practical interest. Likewise, from the work of Reference 4, system stiffness used for external pressure loading of such shell bodies does not affect their initial crippling pressure. As a result, it must be considered that the Tsien criterion of instability is improper or that the large displacement analysis is inapplicable, or both.

These findings, reported in a compendium elsewhere[3], are in agreement with the majority of earlier predictions based upon scanty experimental evidence and qualitative argument but are antithetic to the conclusions reached by Mossakovskii and Smelyi. This direct contradiction, however, merely serves to emphasize the danger in statistical deductions made from an inadequate volume of test data.

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APPENDIX I*

DETERMINATION OF TEST MACHINE EXTENSIONAL RIGIDITY-SERIES C TESTS

The rigidity of the test machines used in this investigation was obtained in accordance with the procedure outlined on page 3 of this report. In this appendix, the actual test data obtained are presented and analyzed. The load deflection histories are presented in Tables I through VIII, and are displayed graphically in Figures 13 through 20. The actual machine stiffnesses are given on the appropriate figures.

* See appendixes 1 and 2 of Reference 2 for additional information.

TABLE I. FORCE-HEAD SEPARATION DATA FOR THE 60,000-LB BALDWIN-LIMA-HAMILTON TESTING MACHINE

Load (lb)	Deflection, Dial No. 1 (inches x 10^4)	Deflection, Dial No. 2 (inches x 10^4)	Average Deflection (inches x 10^4)
85	1.5	1	1.25
140	2.5	2	2.25
195	3.5	3	3.25
265	4.5	4	4.25
350	6.5	6	6.25
385	7	6.5	6.75
475	9	8	8.5
535	10	9	9.5
610	12	10	11
755	13.5	12	12.75
860	15.5	14	14.75
1145	20.5	19	19.75
1240	23	20.5	21.75
1440	26	23	24.5

Plot of data is shown in Figure 13.

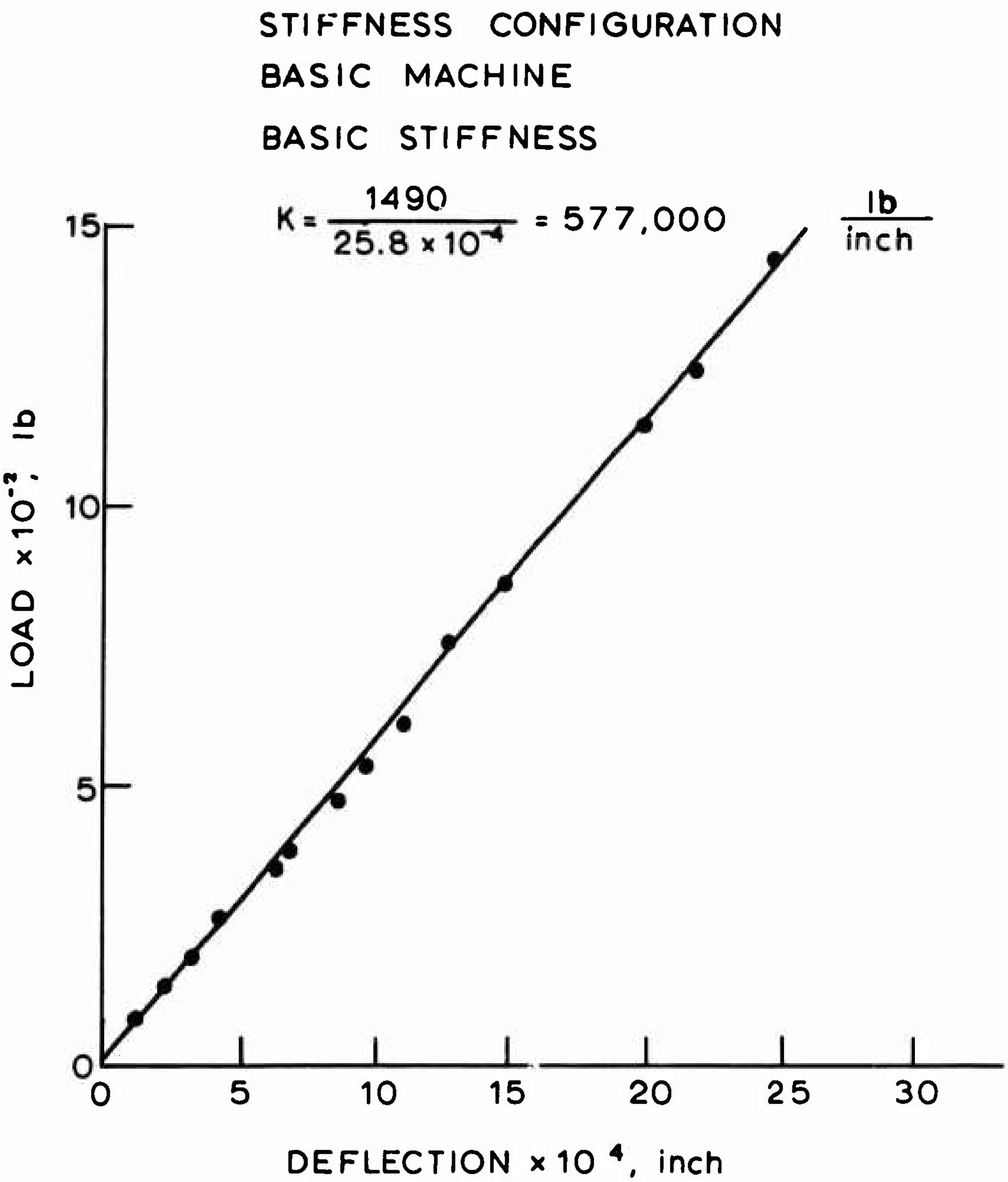

Figure 13. Stiffness Plot of the Basic 60,000-lb Baldwin-Lima-Hamilton Testing Machine - Test Series C and D.

TABLE II. FORCE-HEAD SEPARATION DATA FOR THE 60,000-LB BALDWIN-LIMA-HAMILTON TESTING MACHINE MODIFIED BY 6 LEAF SPRINGS WITH QUARTER-SPAN SUPPORTS

Load (lb)	Deflection, Dial No. 1 (inches x 10^4)	Deflection, Dial. No. 2 (inches x 10^4)	Average Deflection (inches x 10^4)
55	10	0	5
110	17.5	0	8.75
225	24	6	15
280	29	9	19
390	36.5	17	26.75
470	41.5	24	32.75
550	47.5	31.5	39.5
610	50	37	43.5
700	56.5	46.5	51.5
760	59.5	52	55.75
870	67	62.5	64.5
960	71	71	71
1070	77	80.5	78.75
1200	82	92	87
1340	89	104.5	96.75
1490	97	115.5	106.25

Plot of data is shown in Figure 14.

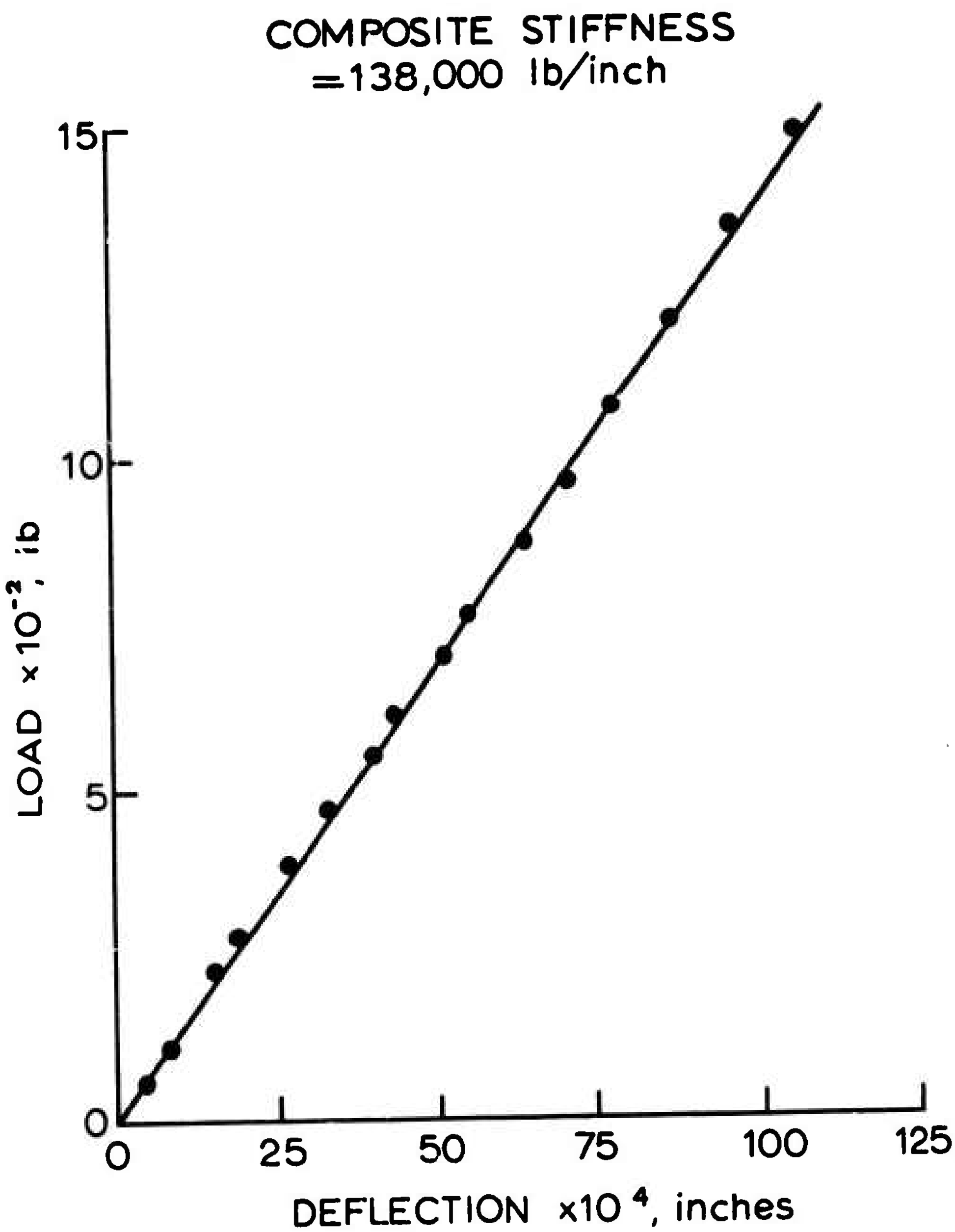

Figure 14. Stiffness Plot of 60,000-lb Baldwin-Lima-Hamilton Testing Machine Modified by Leaf Spring Configuration 62 (With Quarter-Span Supports)- Test Series C and D.

TABLE III. FORCE-HEAD SEPARATION DATA FOR THE 60,000-LB BALDWIN-LIMA-HAMILTON TESTING MACHINE MODIFIED BY 6 LEAF SPRINGS

Load (lb)	Deflection, Dial No. 1 (inches x 10^4)	Deflection, Dial No. 2 (inches x 10^4)	Average Deflection (inches x 10^4)
60	14	8	11
130	27.5	19	23.25
185	38	32	35
245	46	44	45
300	58	58	58
360	67	73	70
450	84	95	89.5
555	101	115	108
635	117	131	124
800	148	162	155
920	170.5	186	178.25
1150	213	232	222.5
1200	224	244	234
1290	240.5	260.5	250.5
1450	272	290.5	281.25

Plot of data is shown in Figure 15.

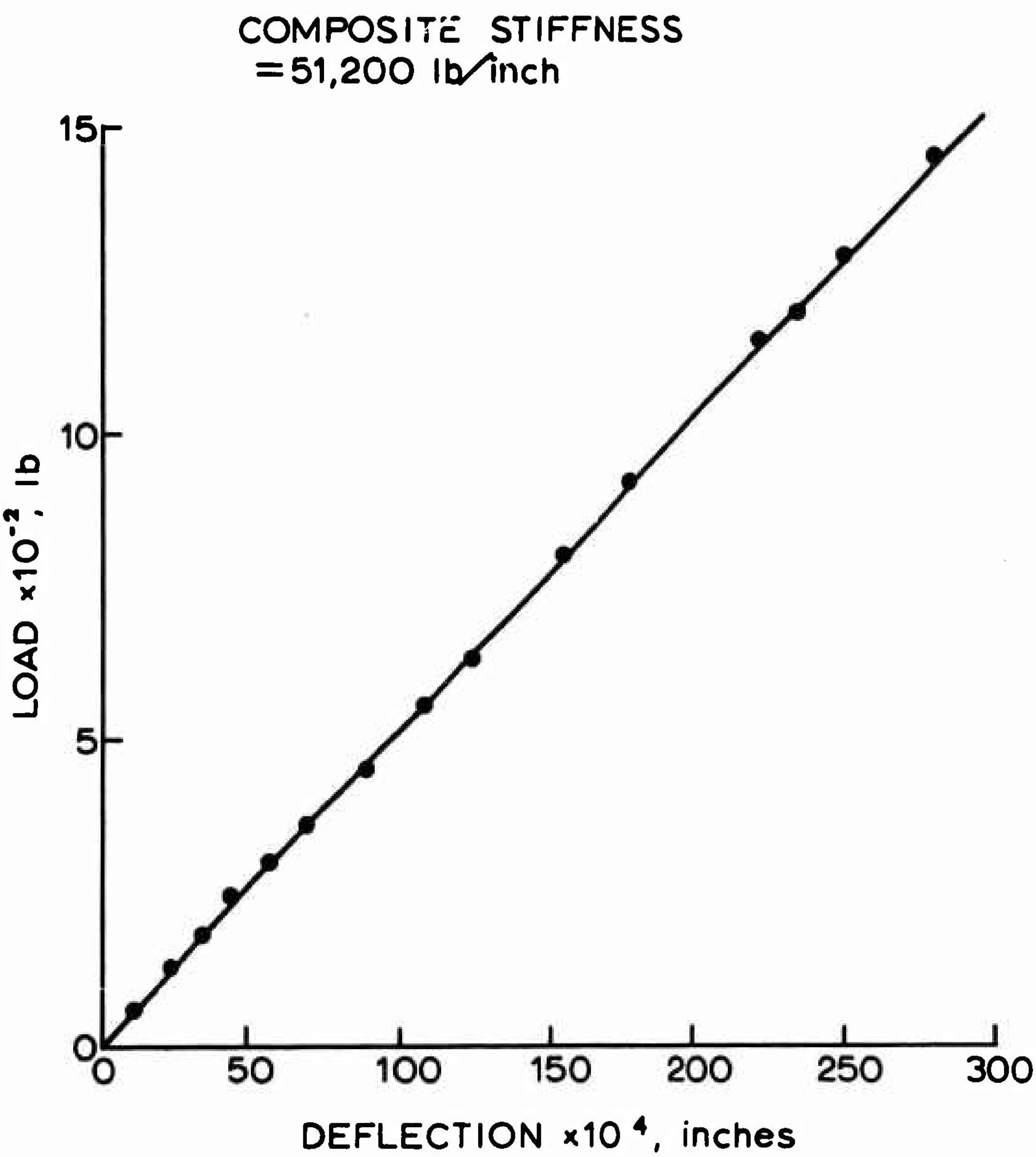

Figure 15. Stiffness Plot of 60,000-lb Baldwin-Lima-Hamilton Testing Machine Modified by Leaf Spring Configuration 61 - Test Series C and D.

TABLE IV. FORCE-HEAD SEPARATION DATA FOR THE 60,000-LB BALDWIN-LIMA-HAMILTON TESTING MACHINE MODIFIED BY 5 LEAF SPRINGS

Load (lb)	Deflection, Dial No. 1 (inches x 10^4)	Deflection, Dial No. 2 (inches x 10^4)	Average Deflection (inches x 10^4)
75	24	16	20
135	34	31	32.5
185	44.5	43	43.5
305	73	72	72.5
400	94	94.5	94
520	121.5	124.5	123
575	133	139	136
630	150	151	150.5
760	184	184	184
805	193	193	193
865	206.5	205.5	206
1100	264	264	264
1230	293	292	292.5
1410	340.5	339.5	340
1505	358	357	357.5

Plot of data is shown in Figure 16.

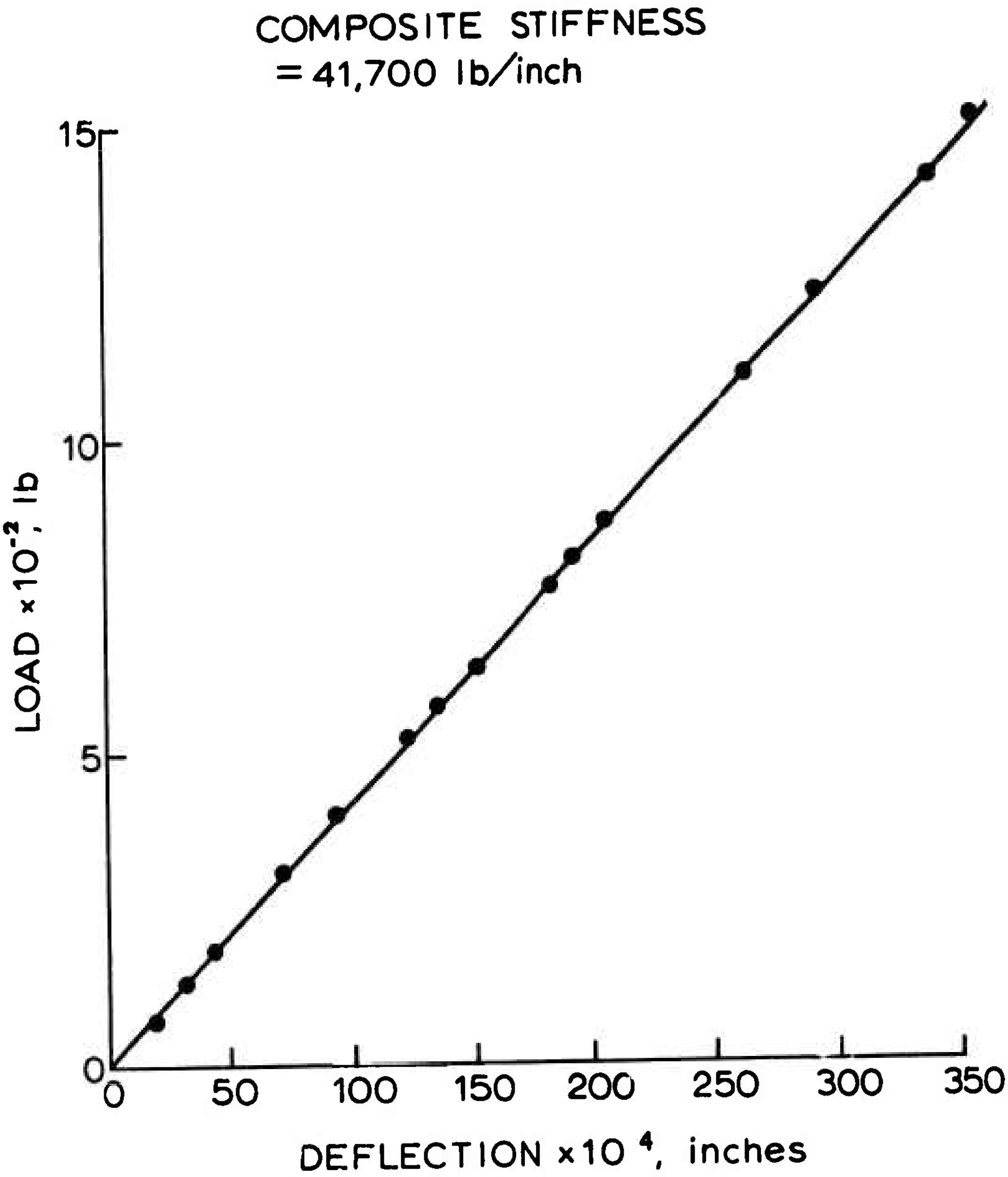

Figure 16. Stiffness Plot of 60,000-lb Baldwin-Lima-Hamilton Testing Machine Modified by Leaf Spring Configuration 51 - Test Series C and D.

TABLE V. FORCE-HEAD SEPARATION DATA FOR THE 60,000-LB BALDWIN-LIMA-HAMILTON TESTING MACHINE MODIFIED BY 4 LEAF SPRINGS

Load (lb)	Deflection, Dial No. 1 (inches x 10^4)	Deflection, Dial No. 2 (inches x 10^4)	Average Deflection (inches x 10^4)
95	27	28	27.5
165	46	48.5	47.2
215	63	67	65
280	85.5	90.5	88
375	114	118	116
470	140	145	142.5
590	178	185	181.5
650	193.5	203	197.8
700	211	221	216
795	237	253.5	245.3
960	290	307	298.5
1070	324	341	332.5
1180	352.5	369	360.8
1270	382	399	390.5
1370	417	433.5	425.3
1480	454.5	470	462.3

Plot of data is shown in Figure 17.

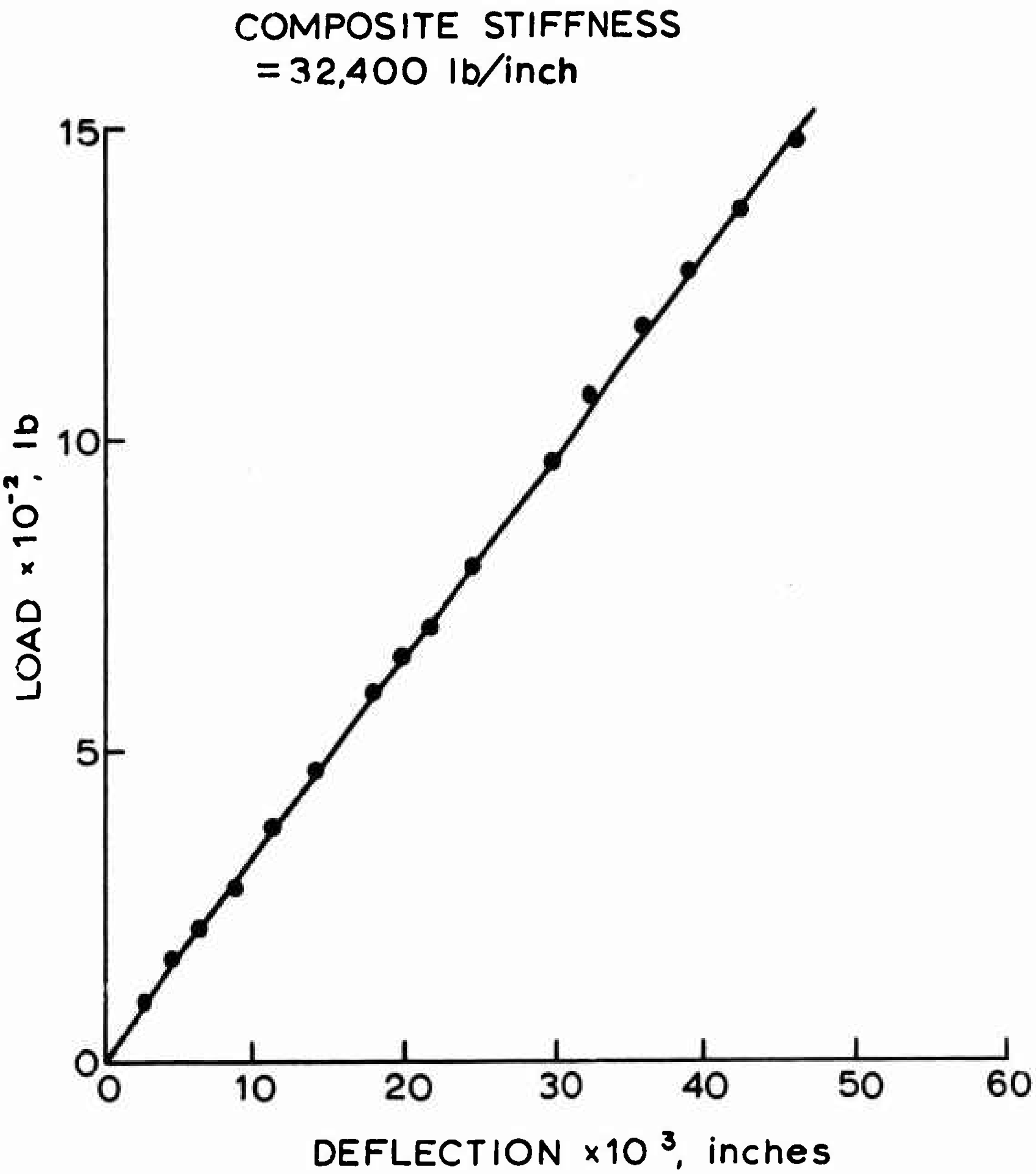

Figure 17. Stiffness Plot of 60,000-lb Baldwin-Lima-Hamilton Testing Machine Modified by Leaf Spring Configuration 41 - Test Series C and D.

TABLE VI. FORCE-HEAD SEPARATION DATA FOR THE 60,000-LB BALDWIN-LIMA-HAMILTON TESTING MACHINE MODIFIED BY 3 LEAF SPRINGS

Load (lb)	Deflection Dial No. 1 (inches x 10^4)	Deflection, Dial No. 2 (inches x 10^4)	Average Deflection (inches x 10^4)
50	18	22	20
130	44	52	48
200	70	87.5	78.8
320	115.5	140.5	128
445	162	190	176
530	193.5	221	207.3
670	247	277	262
780	290	321.5	305.8
920	343	378	360.5
1045	392	428	410
1340	502.5	540	521.3
1440	543	584	563.5
1540	582	624	603

Plot of data is shown in Figure 18.

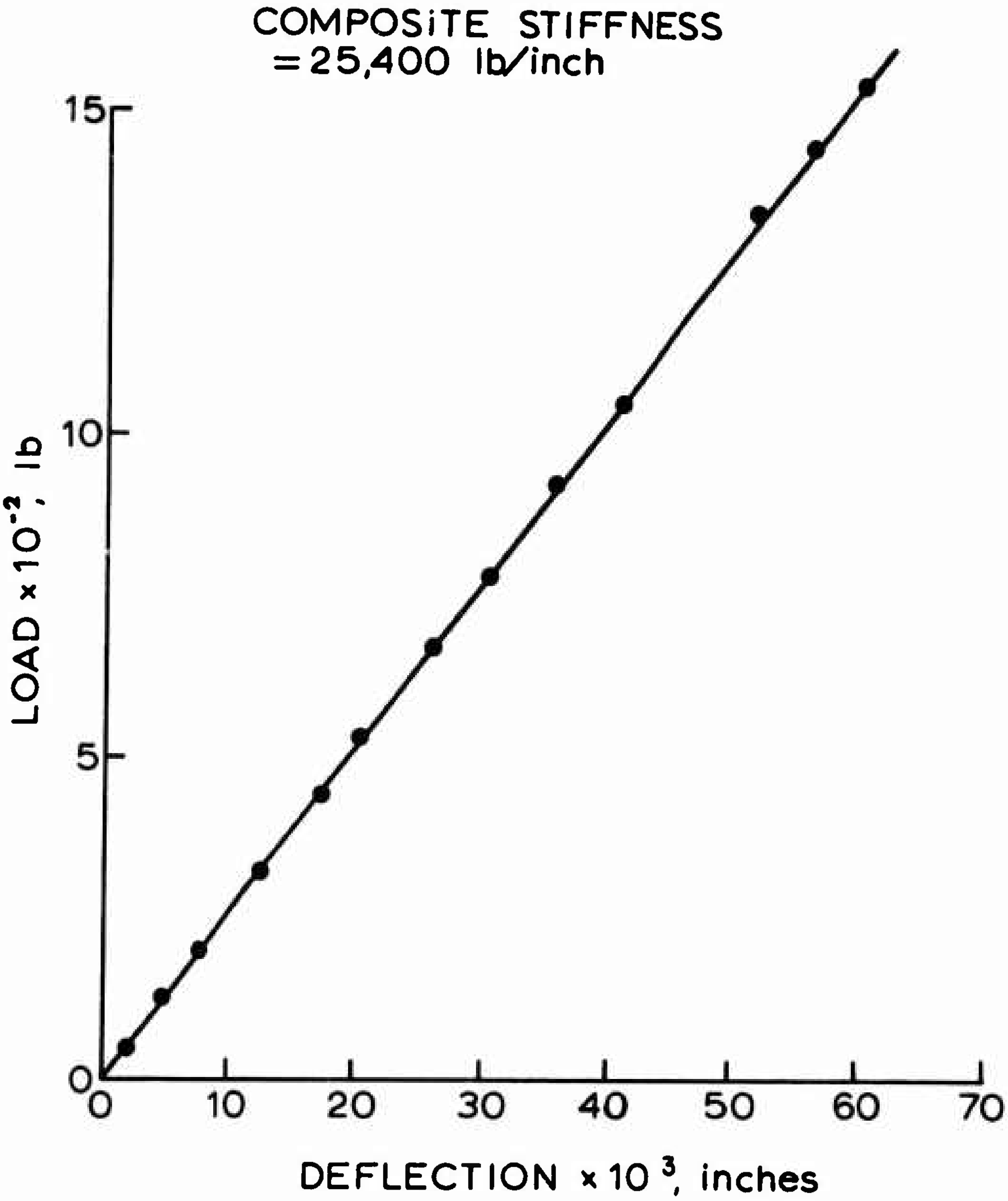

Figure 18. Stiffness Plot of 60,000-lb Baldwin-Lima-Hamilton Testing Machine Modified by Leaf Spring Configuration 31 - Test Series C and D.

TABLE VII. FORCE-HEAD SEPARATION DATA FOR THE 60,000-LB BALDWIN-LIMA-HAMILTON TESTING MACHINE MODIFIED BY 2 LEAF SPRINGS

Load (lb)	Deflection, Dial No. 1 (inches x 10^4)	Deflection, Dial No. 2 (inches x 10^4)	Average Deflection (inches x 10^4)
45	24	26	25
105	63	59	61
210	125.5	129	127.3
290	172.5	167	169.8
380	230.5	229	229.8
470	283.5	286	284.3
585	352	351.5	351.8
695	415	414	414.5
780	471.5	471	471.3
920	551.5	552	551.8
1010	604.5	605.5	605
1170	704	706	705
1300	778	780	779
1440	861	861	861

Plot of data is shown in Figure 19.

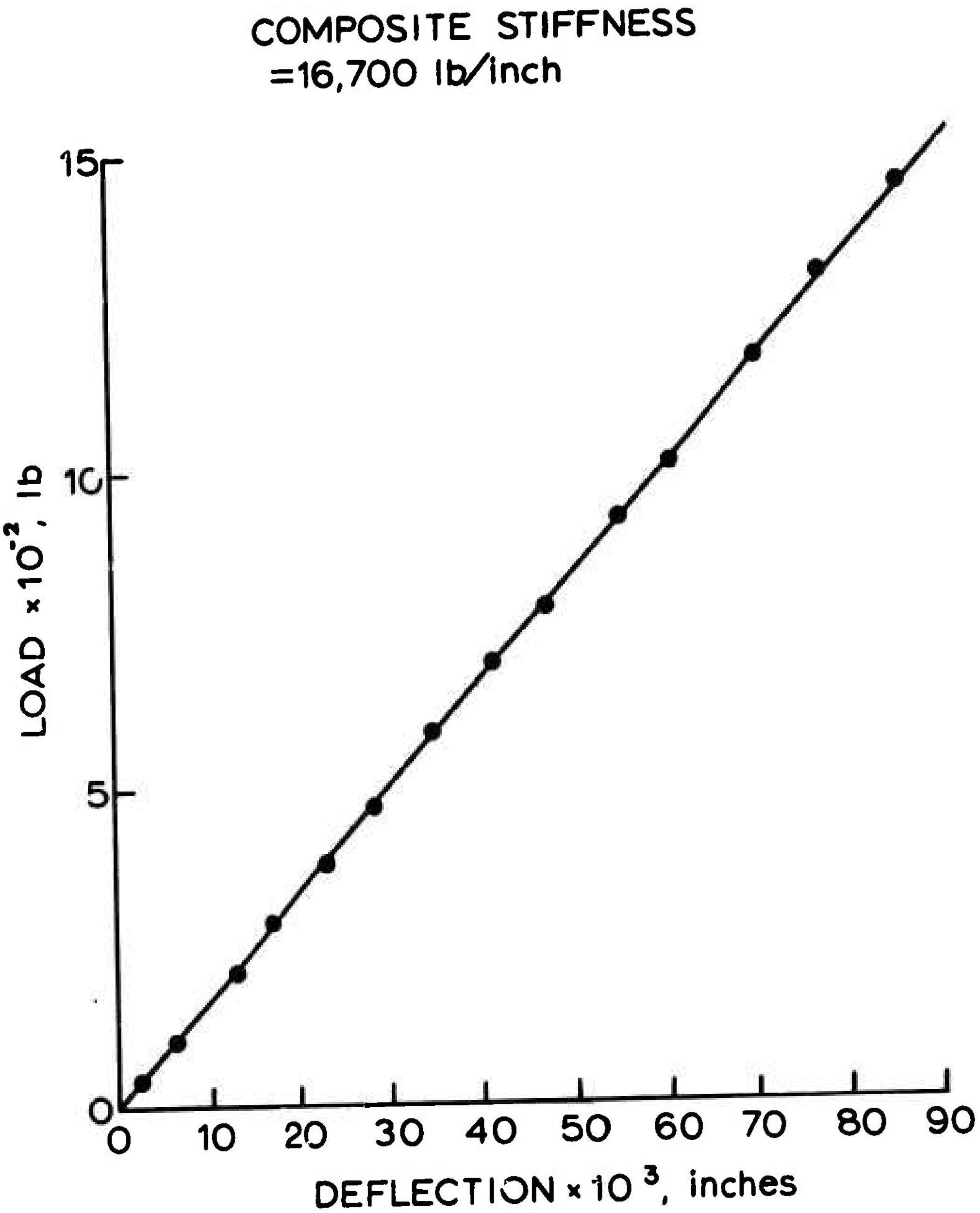

Figure 19. Stiffness Plot of 60,000-lb Baldwin-Lima-Hamilton Testing Machine Modified by Leaf Spring Configuration 21 - Test Series C and D.

TABLE VIII. FORCE-HEAD SEPARATION DATA FOR THE 60,000-LB BALDWIN-LIMA-HAMILTON TESTING MACHINE MODIFIED BY 1 LEAF SPRING

Load (lb)	Deflection Dial No. 1 (inches x 10^3)	Deflection Dial No. 2 (inches x 10^3)	Average Deflection (inches x 10^3)
100	14	12	13
155	21	19	20
215	28	26.5	27.3
305	43	43	43
360	49.5	49.5	49.5
410	55.5	55.5	55.5
480	64.5	64.5	64.5
525	69.5	70	69.7
600	78.5	79.5	79
660	85.5	87	86.7
765	98	100	99
825	105.5	107	106.3
895	114.5	115.5	115
975	124	125.5	124.7
1045	131.5	133.5	132.5
1180	150	151	150.5
1285	162.5	164	163.2
1375	176	177.5	176.7
1560	203.5	204.5	204

Plot of data is shown in Figure 20.

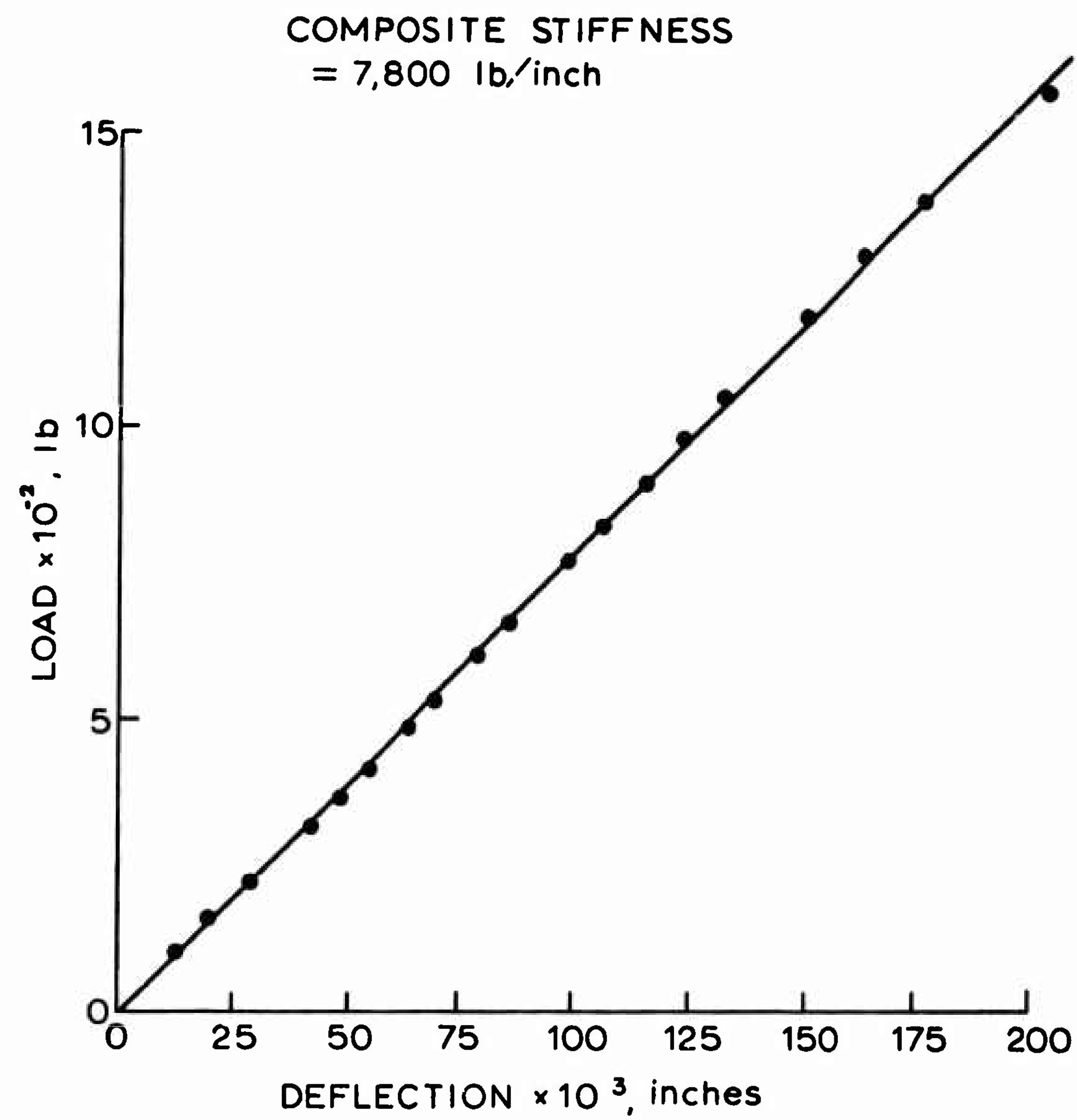

Figure 20. Stiffness Plot of 60,000-lb Baldwin-Lima-Hamilton Testing Machine Modified by Leaf Spring Configuration 11 - Test Series C and D.

APPENDIX II
STATISTICAL ANALYSIS OF THE EXPERIMENTAL DATA OBTAINED IN THE SERIES C TESTS

The test data obtained in the series C tests are presented and analyzed in this appendix. Eight levels of machine stiffness were used in this series. These stiffnesses were obtained as described in Section 5; the values are computed in Appendix I and listed in Table IX together with the corresponding buckling loads obtained from the repeated tests.

A linear regression analysis is made on these data.[10] For this purpose, a regression line is chosen having the form

$$y = \alpha + \beta\,(x - \bar{x})$$

$$y = A' + \beta\,(x)$$

where

y = buckling load, lb

x = machine stiffness, lb/in.

The parameters α and β are estimated by the method of least squares to obtain the empirical regression line

$$Y = a + b\,(x - \bar{x}) = A + bx$$

From the data presented in Table X, the mean buckling load may be computed as

$$A = \bar{y} = \frac{\sum_{i=1}^{n} \bar{y}_i}{n} = \frac{3653.8}{8} = 456.72 \text{ lbs.}$$

while the mean value of machine stiffness used is given by

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{890200}{8} = 111{,}275 \text{ lb/in.}$$

The slope b is defined by

$$b = \frac{k\sum_{i=1}^{n} (x_i - \bar{x})(\bar{y}_i - \bar{y})}{k\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

and for this case is given as

$$b = -\frac{572057}{259,310,973,000} = -0.2206 \times 10^{-5} \text{in.}$$

Thus, the empirical regression line becomes

$$Y = 456.72 - 0.2206 \times 10^{-5} (x - \bar{x})$$

This line is drawn in Figure 12 through the discrete data points, $\bar{y}_i$.

Since more than one value of buckling load was determined at each stiffness, the hypothesis regarding the linearity of the regression curve may be tested by comparing the variation about the regression line with that existing within sets. Linearity is rejected if

$$F = \frac{s_2^2}{s_1^2} \geq F_\alpha;\ n-2,\ n(k-1)$$

where n = number of sets

k = number of values within each set

$$s_2^2 = k \sum_{i=1}^{n} (\bar{y}_i - Y_i)^2 / (n-2) \quad \text{(variance about regression line)}$$

$$s_1^2 = \sum_{i=1}^{n} \sum_{\eta=1}^{k} (y_{i\eta} - \bar{y}_i)^2 / n(k-1) \quad \text{(variance within sets)}$$

$F_\alpha;\ n-2,\ n(k-1)$ = the 100 α percentage point of the F distribution with $(n-2)$ and $n(k-1)$ degrees of freedom

$y_{i\eta}$ = the η th value in the i th set

$\bar{y}_i$ = the average value of the i th set

Y = the empirical value of y corresponding to x_i

It is evident that $\sum_{i=1}^{n} (\bar{y}_i - Y_i)^2$ is just the sum of squares of the deviations of the average value of each set about the fitted line. The computations are carried out in Table X using the relation

$$\sum_{i=1}^{n} (\bar{y}_i - Y_i)^2 = \sum_{i=1}^{n} (\bar{y}_i - \bar{y})^2 - \frac{\left[\sum_{i=1}^{n} (x_i - \bar{x})(\bar{y}_i - \bar{y})\right]^2}{\sum_{i=1}^{n} (x_i = \bar{x})^2}$$

Thus, variance of y about the mean $\alpha + b(x - \bar{x})$, described exactly by σ_2^2, is estimated from the data as

$$s_2^2 = 6.28 - \frac{(572,057)^2}{259,310,973,000} = 5.02 \text{ lb}^2$$

Likewise, it is evident that $\sum_{i=1}^{r} \sum_{\eta=1}^{k} (y_{i\eta} - \bar{y}_i)^2$ is the sum for all sets of the sum of the squares of the deviations of the individual values within each set about the mean value of the set. The computations are carried out in Table X using the relation

$$\sum_{i=1}^{n} \sum_{\eta=1}^{k} (y_{i\eta} - \bar{y})^2 = \sum_{i=1}^{n} \sum_{\eta=1}^{k} {y_{i\eta}}^2 - \sum_{i=1}^{n} \sum_{\eta=1}^{k} (y_{i\eta})^2/k_i$$

Before computing s_1^2, the homogeneity of the several set variances is tested by means of Cochran's test.[11] The hypothesis of equality is accepted if

$$g = \frac{\text{largest } s_i^2}{\sum_{i=1}^{n} s_i^2} \leq g_\alpha$$

where

$$s_i^2 = \sum_{\eta=1}^{k} (y_{i\eta} - \bar{y}_i)^2 / (k_i - 1)$$

is the ith set variance and g_α depends upon the level of significance α, n and k.[11] Choosing α = 5 percent, $g_{\alpha(6,8)}$ is 0.360. Then, from Table IX,

$$g = \frac{8.8}{43.6} = 0.202 < 0.360$$

Thus, the eight variances do not differ significantly and may be combined to give the single estimate

$$s_1^2 = \frac{218}{40} = 5.45 \text{ lb.}^2$$

It is found that

$$F_{\alpha;\, n-2,\, n(k-1)} = F_{0.05;6,40} = 2.34$$

Substitution yields

$$F = \frac{5.02}{5.45} = 0.921$$

Thus, the hypothesis of linearity is accepted at the 95-percent level. This leads to a pooled estimate of variance about the regression line:

$$s^2 = \frac{k\sum_{i=1}^{n}(\bar{y}_i - Y_i)^2 + \sum_{i=1}^{n}\sum_{\eta=1}^{k}(y_{i\eta} - \bar{y}_i)^2}{nk - 2}$$

$$s^2 = \frac{6(5.02) + 218}{46} = 5.3939$$

and a standard deviation:

$$s = 2.3225$$

The variance of the estimate b of the slope and the estimate A of the intercept are normally distributed with means β and A', respectively. The variance of b is given by

$$\sigma_b^2 = \frac{\sigma^2}{k\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

From Table X and the estimate for σ^2, we may write

$$\sigma_b^2 = \frac{5.3939}{6(259,310,973,000)} = 3.4668 \times 10^{-12}$$

Thus, the standard deviation is

$$\sigma_b = 0.1862 \times 10^{-5}$$

The variance of A is given by

$$\sigma_A^2 = \sigma^2 \left[\frac{1}{kn} + \frac{\bar{x}^2}{k\sum_{i=1}^{n} (x_i - \bar{x})^2} \right]$$

and by substituting appropriate values from Table X, we arrive at

$$\sigma_A^2 = 5.3939 \left[\frac{1}{48} + \frac{12,382,125,600}{6(259,310,973,000)} \right] = 0.155299$$

and

$$\sigma_A = 0.3940$$

In addition to the point estimates of slope and intercept, confidence intervals for β and A' may be established with confidence coefficient $1 - \alpha$. They are given for β by

$$b \pm t_{\alpha/2;kn - 2} \sigma_b$$

and for A' by

$$A \pm t_{\alpha/2;kn - 2} \sigma_A$$

where $t_{\alpha/2;kn - 2}$ is the 100 $\alpha/2$ percentage point of Student's t distribution. Choosing α = 5 percent, the 95-percent confidence interval estimate for β is

$$b \pm t_{.025;46} \sigma_b = \left[-0.2206 \times 10^{-5} \pm 2.015(0.1862) \times 10^{-5}\right]$$

$$= \left[-0.5958;\ 0.1546\right] \times 10^{-5} \text{ in.}$$

The confidence interval for A' is

$$A \pm t_{.025;\,46}\,\sigma_A = \left[\, 456.97 \pm 2.015(0.6996) \,\right]$$

$$= \left[\, 455.56;\ 458.38 \,\right] \text{ lb}$$

The theoretical slope β and the intercerpt A' are examined statistically to check the initial assumption that buckling load is a linear function of machine rigidity. The significance tests for these coefficients are given in Table XI.

The empirical regression line shown in Figure 12 is almost horizontal. In addition, the 95-percent confidence interval for β includes the possibility of a zero slope. This suggests that there is no relationship between x (machine stiffness) and the mean value of y (mean buckling load), and that the small empirical slope b is due to accidental variation of the data.

This hypothesis is tested by putting $\beta = 0$ in Table XI . The test statistic is

$$t = \frac{b}{\sigma_b} = -\frac{0.2206}{0.1862} = -1.185$$

At the 5-percent level of significance, the criterion for rejection becomes

$$|t| \geq t_{\alpha/2;kn-2} = t_{0.025;46} = 2.015$$

Hence

$$1.185 < 2.015$$

The hypothesis that the theoretical slope is zero is accepted at the 95-percent level.

A measure of the sensitivity of the analysis is obtained from an examination of the OC curve at this level. For a sample size of 48, there is a 95-percent probability of detecting a value of d = 0.6.

For this test,

$$d = \frac{|\beta_o - \beta_1|}{\sigma_b\sqrt{kn-1}} = 0.6$$

Hence,

$$|\beta_1| = 0.6\,(0.1862 \times 10^{-5})\,\sqrt{47}$$

or

$$|\beta_1| = 0.7659 \times 10^{-5}$$

Thus, the hypothesis $-\beta = 0$ would be rejected at the 95-percent level if it differed from zero by as little as 0.7659×10^{-5} in. A slope of this magnitude may be seen in better perspective if we note that it implies a change at zero stiffness given by

$$\Delta_y = \beta_1 \bar{x}$$

$$= (0.7659 \times 10^{-5})(111{,}275)$$

$$= 0.85 \text{ lb}$$

which is only

$$(100)\frac{0.85}{456.72} = 0.23\%$$

of the average critical load.

The assumption that no relationship exists between x and the mean value of y may be tested further by assuming that the theoretical intercept A' is equal to the mean value of the buckling load. The 95-percent confidence interval for A' includes the possibility of a value equal to $\bar{y}$. To test this hypothesis, A'_o is set equal to $\bar{y}=456.72$ in Table XI. The test statistic is

$$t = \frac{A - A'_o}{\sigma_A}$$

$$t = \frac{\left[456.72 + (0.2206 \times 10^{-5})\ (111{,}275) - 456.72\right]}{0.3940}$$

$$t = \frac{0.2455}{0.3940} = 0.6231$$

Since $$0.623 < t_{.025;46} = 2.015$$

the hypothesis $-A' = \bar{y}$ is accepted at the 95-percent level.

Then, from the OC curve,

$$d = \frac{|A'_o - A'_1|}{\sigma_A \sqrt{kn-1}} = 0.6$$

Hence, $$|A'_o - A'_1| = 0.6(0.3940)\sqrt{47}$$

$$|A'_o - A'_1| = 1.62 \text{ lb}$$

Thus, the hypothesis $-A' = \bar{y}$ would be rejected at the 95-percent level if the theoretical intercept differed from the mean value by as little as 1.62 lb, or only $[100(1.62)/456.72] = 0.35\%$ of the average critical load.

TABLE IX. LINEAR REGRESSION ANALYSIS - SERIES C RESULTS

Test No. i	Composite Stiffness k, lb/in. x_i	Initial Buckling Load P_{cr}, lb $y_{i\eta}$: y_{i1}	y_{i2}	y_{i3}	y_{i4}	y_{i5}	y_{i6}	k_i	S_i	SS_i	S_i^2/k_i	SSD_i	s_i^2
1	577,000	459	455	454	458	454	454	6	2,784	1,245,818	1,245,793	25	5.0
2	138,000	453	460	455	455	460	455	6	2,738	1,249,484	1,249,440	44	8.8
3	51,200	455	458	458	460	459	460	6	2,750	1,260,434	1,260,417	17	3.4
4	41,700	456	455	457	459	459	456	6	2,742	1,253,108	1,253,094	14	2.8
5	32,400	456	456	459	454	457	452	6	2,734	1,245,822	1,245,793	29	5.8
6	25,400	459	454	459	455	460	460	6	2,747	1,257,703	1,257,668	35	7.0
7	16,700	454	453	458	455	460	457	6	2,737	1,248,563	1,248,528	35	7.0
8	7,800	458	458	454	459	455	457	6	2,741	1,252,199	1,252,180	19	3.8
$\sum$	890,200							48	21,923	10,013,131	10,012,913	218	43.6

$S_i = \sum_{\eta=1}^{k} y_{i\eta}$ $\quad SSD_i = \sum_{\eta=1}^{k} (y_{i\eta} - \bar{y}_i)^2$ $\quad SS_i = \sum_{\eta=1}^{k} y^2_{i\eta}$ $\quad s_i^2 = SSD_i/(k_i-1)$

TABLE X. LINEAR REGRESSION ANALYSIS - SERIES C TESTS

Test No.	Leaf Spring Configuration No. Springs	Leaf Spring Configuration Support Position	Composite Stiffness K lb/in. (x_i)	Initial Buckling Load $\bar{P}_{cr}$ lb $(\bar{y}_i)^a$	$(x_i-\bar{x})$	$(x_i-\bar{x})^2$	$(\bar{y}_i-\bar{y})$	$(x_i-\bar{x})x(\bar{y}_i-\bar{y})$	$(\bar{y}_i-\bar{y})^2$
1	(Basic Machine)		577,000	455.7	465,725	216,699,780	-1.0	-465,725	1.00
2	6	(2)[b]	138,000	456.3	26,725	714,226	-0.4	- 10,690	.16
3	6	1	51,200	458.3	-60,075	3,609,000	1.6	- 96,120	2.56
4	5	1	41,700	457.0	-69,575	4,840,680	0.3	- 20,873	.09
5	4	1	32,400	455.7	-78,875	6,221,266	-1.0	78,875	1.00
6	3	1	25,400	457.8	-85,875	7,374,516	1.1	-94,463	1.21
7	2	1	16,700	456.2	-94,575	8,944,430	-0.5	47,287	.25
8	1	1	7,800	456.8	-103,475	10,707,075	0.1	-10,348	.01
$\sum$			890,200	3653.8		259,310,973 x 10^3		-572,057	6.28

$\bar{x}$ = 111,275

$\bar{y}$ = 456.7

(a) Reference Table IX, $\bar{y}_i = S_i/k_i$

(b) Reference Figure 5(c)

TABLE XI. SIGNIFICANCE TEST FOR SLOPE AND INTERCEPT OF STRAIGHT LINE (See Reference 11)

Hypothesis	Test Statistic t	Criteria for Rejection $\lvert t \rvert \geq$	Operating Characteristic Abscissa Value d
$\beta = \beta_o$	$\dfrac{b - \beta_o}{\sigma_b}$	$t_{\alpha/2;kn - 2}$	$\dfrac{\lvert \beta_o - \beta_1 \rvert}{\sigma_b \sqrt{kn - 1}}$
$A' = A'_o$	$\dfrac{A - A'_o}{\sigma_A}$	$t_{\alpha/2;kn - 2}$	$\dfrac{\lvert A'_o - A'_1 \rvert}{\sigma_A \sqrt{kn - 1}}$

n = number of sets

k = number of values within each set

α = level of significance

β_o and A'_o = hypothetical values of slope and intercept, respectively

β_1 and A'_1 = the variations of slope and intercept from the values which can be detected for the given sample size and chosen probability. In finding the value of d, the (n - 1) curve in Reference 11 is used.

APPENDIX III
PRESENTATION AND STATISTICAL ANALYSIS OF THE EXPERIMENTAL DATA OBTAINED IN THE SERIES D TESTS

The test data obtained in the series D tests are presented and analyzed in this appendix. Eight levels of machine stiffness were used in this series. These stiffnesses were obtained as described on page 3 ; the values are computed in Appendix I and are listed in Table XII together with the corresponding buckling loads obtained from the repeated tests.

A linear regression analysis is made on these data.[10] For this purpose, a regression line is chosen having the form

$$y = \alpha + \beta\,(x - \bar{x})$$

$$y = A' + \beta\,(x)$$

where

$$y = \text{buckling load, lb}$$

$$x = \text{machine stiffness, lb/in.}$$

The parameters α and β are estimated by the method of least squares to obtain the empirical regression line

$$Y = a + b\,(x - \bar{x}) = A + bx$$

From the data presented in Table XIII, the mean buckling load may be computed as

$$a = \bar{y} = \frac{\sum_{i=1}^{n} \bar{y}_i}{n} = \frac{900.7}{8} = 112.59 \text{ lb}$$

while the mean value of machine stiffness used is given by

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{8} = \frac{890200}{8} = 111{,}275 \text{ lb/in.}$$

The slope b is defined by

$$b = \frac{k\sum_{i=1}^{n} (x_i - \bar{x})(\bar{y}_i - \bar{y})}{k\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

and for this case is given as

$$b = -\frac{611{,}344}{259{,}310{,}973{,}000} = -0.23576 \times 10^{-5} \text{ in.}$$

Thus, the empirical regression line becomes

$$Y = 112.59 - 0.23576 \times 10^{-5}(x - \bar{x})$$

This line is drawn in Figure 13 through the discrete data points, $\bar{y}_i$.

Since more than one value of buckling load was determined at each stiffness, the hypothesis regarding the linearity of the regression curve may be tested by comparing the variation about the regression line with that existing within sets. Linearity is rejected if

$$F = \frac{s_2^2}{s_1^2} > F_{\alpha;\, n-2,\, n(k-1)}$$

where

n = number of sets

k = number of values within each set

$$s_2^2 = k \sum_{i=1}^{n} (\bar{y}_i - Y_i)^2 / (n - 2) \quad \text{(variance about regression line)}$$

$$s_1^2 = \sum_{i=1}^{n} \sum_{\eta=1}^{k} (y_{i\eta} - \bar{y}_i)^2 / n(k - 1) \quad \text{(variance within sets)}$$

$F_{\alpha;\, n - 2,\, n(k - 1)}$ = the 100 α percentage point of the F distribution with $(n - 2)$ and $n(k - 1)$ degrees of freedom

$y_{i\eta}$ = the ηth value of the i th set

$\bar{y}_i$ = the average value of the i th set

Y_i = the empirical value of y corresponding to x_i

It is evident that

$$\sum_{i=1}^{n} (\bar{y}_i - Y_i)^2$$

is just the sum of squares of the deviations of the average value of each set about the fitted line. The computations are carried out in Table XIII using the relation

$$\sum_{i=1}^{n} (\bar{y}_i - Y_i)^2 = \sum_{i=1}^{n} (\bar{y}_i - \bar{y})^2 - \frac{\left[\sum_{i=1}^{n} (x_i - \bar{x})(\bar{y}_i - \bar{y})\right]^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

Thus, the variance of y about the mean $\alpha + \beta(x - \bar{x})$, described exactly by σ^2, is estimated from the data as

$$s_2^2 = 7.89 - \frac{(611,344)^2}{259,310,973,000} = 6.45 \text{ lb}^2$$

Likewise, it is evident that

$$\sum_{i=1}^{n} \sum_{\eta=1}^{k} (y_{i\eta} - \bar{y}_i)^2$$

is the sum for all sets of the sum of the squares of the deviations of the individual values within each set about the mean value of the set. The computations are carried out in Table XII using the relation

$$\sum_{i=1}^{n} \sum_{\eta=1}^{k} (y_{i\eta} - \bar{y}_i)^2 = \sum_{i=1}^{n} \sum_{\eta=1}^{k} {y_i}^2 - \sum_{i=1}^{n} \sum_{\eta=1}^{k} (y_{i\eta})^2/k_i$$

Before computing s_1^2, the homogeneity of the several set variances is tested by means of Cochran's test.[11] The hypothesis of equality is accepted if

$$g = \frac{\text{largest } s_1^2}{\sum_{i=1}^{n} s_i^2} \leq g_\alpha$$

where

$$s_i^2 = \sum_{\eta=1}^{k} (y_{i\eta} - \bar{y}_i)^2 / (k_i - 1)$$

is the i th set variance and g_α depends upon the level of significance α, n and k.[11] Choosing α = 5 percent, $g_{\alpha(6,8)}$ is 0.360. Then, from Table XII,

$$g = \frac{6.2}{29.2} = 0.212 < 0.360$$

Thus, the eight variances do not differ significantly and may be combined to give the single estimate

$$s_1^2 = \frac{146}{40} = 3.65$$

It is found that

$$F_{\alpha;n-2,\ n(k-1)} = F_{0.05;6,60} = 2.34$$

Substitution yields

$$F = \frac{6.45}{3.65} = 1.76 < 2.34$$

Thus, the hypothesis of linearity is accepted at the 95-percent level. This leads to a pooled estimate of variance about the regression line:

$$s^2 = \frac{k\sum_{i=1}^{n} (\bar{y}_i - Y_i)^2 + \sum_{i=1}^{n} \sum_{\eta=1}^{k} (y_{i\eta} - \bar{y}_i)^2}{nk - 1}$$

$$s^2 = \frac{6(6.45) + 146}{46} = 4.0152$$

and a standard deviation:

$$s = 2.004 \text{ lb}$$

The variance of the estimate b of the slope and the estimate A of the intercept are normally distributed with means β and A', respectively. The variance of b is given by

$$\sigma_b^2 = \frac{\sigma^2}{k\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

From Table XIII and the estimate for σ^2, we may write

$$\sigma_b^2 = \frac{4.0152}{6(259,310,973,000)} = 2.580685 \times 10^{-12}$$

Thus, the standard deviation is given as

$$\sigma_b = 0.1606 \times 10^{-5}$$

The variance of A is given by

$$\sigma_A^2 = \sigma^2 \left[\frac{1}{nk} + \frac{\bar{x}^2}{k\sum_{i=1}^{n} (x_i - \bar{x})^2} \right]$$

and by substituting appropriate values from Table XIII we arrive at

$$\sigma_A^2 = 4.0152 \left[\frac{1}{48} + \frac{12,382,125,600}{6(259,310,973,000)} \right] = 0.115604$$

Then,

$$\sigma_A = 0.3400$$

In addition to the point estimates of slope and intercept, confidence intervals for β and A' may be established with confidence coefficient $1 - \alpha$. They are given for β by

$$b \pm t_{\alpha/2;\ kn - 2}\sigma_b$$

and for A' by

$$A \pm t_{\alpha/2;\ kn - 2}\sigma_A$$

where $t_{\alpha/2;\ kn - 2}$ is the 100 $\alpha/2$ percentage point of Student's t distribution. Choosing α = 5 percent, the 95-percent confidence interval estimate for β is

$$b \pm t_{.025;\ 46}\sigma_b = \left[-0.2358 \times 10^{-5} \pm 2.015(0.1606) \times 10^{-5}\right]$$

$$= \left[-0.559;\ 0.088\right] \times 10^{-5} \text{ in.}$$

The confidence interval for A' is

$$A \pm t_{.025;\ 46}\sigma_A = \left[112.85 \pm 2.015(0.3400) \right]$$

$$= \left[112.17;\ 113.53 \right] \text{lb}$$

The theoretical slope β and the intercept A' are examined statistically to check the initial assumption that buckling load is a function of machine rigidity. The significance tests for these coefficients are given in Table XI.

The empirical regression line shown in Figure 13 is almost horizontal. In addition, the 95-percent confidence interval for β includes the possibility of a zero slope. This suggests that there is no relationship between x (machine stiffness) and the mean value of y (mean buckling load), and that the small empirical slope b is due to accidental variation of the data.

This hypothesis is tested by putting $\beta_o = 0$ in Table XI. The test statistic is

$$t = \frac{b}{\sigma_b} = - \frac{0.23476}{0.1606} = -1.468$$

At the 5-percent level of significance, the criterion for rejection becomes

$$|t| \geq t_{\alpha/2;kn-2} = t_{0.025;\ 46} = 2.015$$

Since

$$1.468 < 2.015$$

the hypothesis that the theoretical slope is zero is accepted at the 95-percent level.

A measure of the sensitivity of the analysis is obtained from an examination of the OC curve at this level. For a sample size of 48, there is a 95-percent probability of detecting a value of d = 0.6.

For this test,

$$d = \frac{|\beta_o - \beta_1|}{\sigma_b \sqrt{kn - 1}} = 0.6$$

Hence,

$$|\beta_1| = 0.6\ (0.1606 \times 10^{-5})\ \sqrt{47}$$

$$|\beta_1| = 0.6606 \times 10^{-5}$$

Thus, the hypothesis $-\beta = 0$ would be rejected at the 95-percent level if it differed from zero by as little as 0.6606×10^{-5} in. A slope of this magnitude may be seen in better perspective if we note that it implies a change at zero stiffness given by

$$\Delta y = \beta_1 \bar{x}$$

$$= (0.6606 \times 10^{-5})\ (111275)$$

$$= 0.74 \text{ lb}$$

which is only

$$(100)\ \frac{0.74}{11.259} = 0.66\%$$

of the average critical load.

The assumption that no relationship exists between x and the mean value of y may be tested further by assuming that the theoretical intercept A' is equal to the mean value of the buckling load. The 95-percent confidence interval for A' includes the possibility of a value equal to $\bar{y}$. To test this hypothesis, A'_o is set equal to $\bar{y} = 112.59$ in Table XI. The test statistic is

$$t = \frac{A - A'_o}{\sigma_A}$$

$$t = \frac{\left[112.59 + (0.23576 \times 10^{-5})\ (111275) - 112.59\right]}{0.3400}$$

$$t = \frac{0.2623}{0.3400} = 0.7715$$

Since

$$0.7715 < t_{.025;46} = 2.015$$

the hypothesis $-A' = \bar{y}$ is accepted at the 95-percent level.

Then, from the OC curve,

$$d = \frac{|a'_o - A'_1|}{\sigma_A \sqrt{kn} - 1} = 0.6$$

Hence,

$$|A'_o - A'_1| = 0.6(0.3400)\ \sqrt{47}$$

$$|A'_o - A'_1| = 1.40 \text{ lb}$$

Thus, the hypothesis $-A' = \bar{y}$ would be rejected at the 95-percent level if the theoretical intercept differed from the mean value by as little as 1.40 lb or only

$$\frac{100(1.40)}{112.59} = 1.24\%$$

of the average critical load.

TABLE XII. LINEAR REGRESSION ANALYSIS - SERIES D RESULTS

Test No. i	Composite Stiffness K, lb./in. x_i	Initial Buckling Load P_{cr}, lb. $y_{i\eta}$: y_{i1}	y_{i2}	y_{i3}	y_{i4}	y_{i5}	y_{i6}	k_i	S_i	SS_i	S_i^2/k_i	SSD_i	s_i^2
1	577, [illegible]0	112	112	112	109	110	113	6	668	74,382	74,371	11	2.2
2	138,000	112	115	109	112	113	116	6	677	76,419	76,388	31	6.2
3	51,200	113	112	112	114	116	114	6	681	77,305	77,294	11	2.2
4	41,700	112	115	118	112	114	114	6	685	78,229	78,204	25	5.0
5	32,400	110	111	114	111	115	115	6	676	76,188	76,163	25	5.0
6	25,400	116	111	114	112	115	112	6	680	77,086	77,067	19	3.8
7	16,700	111	109	110	112	113	112	6	667	74,159	74,148	11	2.2
8	7,800	113	111	113	112	109	113	6	671	75,053	75,040	13	2.6
$\sum$	890,200							48	5,405	608,821	608,675	146	29.2

$S_i = \sum_{\eta=1}^{k} y_{i\eta}$ $\quad SS_i = \sum_{\eta=1}^{k} y_{i\eta}^2$ $\quad SSD_i = \sum_{\eta=1}^{k} (y_{i\eta} - \bar{y}_i)^2$ $\quad s_i^2 = SSD_i/(k_i-1)$

TABLE XIII. LINEAR REGRESSION ANALYSIS - SERIES D RESULTS

Test No.	Leaf Spring Configuration No. Springs	Support Position	Composite Stiffness K lb/in (x_i)	Initial Buckling Load P_{cr} lb. $(\bar{y}_i)$[a]	$(x_i - \bar{x})$	$(x_i - \bar{x})^2$ x 10^3	$(\bar{y}_i - \bar{y})$	$(x_i - \bar{x})$x $(\bar{y}_i - \bar{y})$	$(\bar{y}_i - \bar{y})^2$
1	(Basic Machine)		577,000	111.3	465,725	216,899,780	-1.3	-605,443	1.69
2	6	(2)[b]	138,000	112.8	26,725	714,226	0.2	5,345	0.04
3	6	1	51,200	113.5	-60,075	3,609,000	0.9	- 54,067	0.81
4	5	1	41,700	114.1	-69,575	4,840,680	1.5	-104,363	2.25
5	4	1	32,400	112.7	-78,875	6,211,266	0.1	- 7,888	0.01
6	3	1	25,400	113.3	-85,875	7,374,516	0.7	- 60,113	0.49
7	2	1	16,700	111.2	-94,575	8,944,430	-1.4	132,405	1.96
8	1	1	7,800	111.8	-103,475	10,707,075	-0.8	82,780	0.64
$\sum$			890,200	900.7		259,310,973 x 10^3		-611,344	7.89

$\bar{x}$ = 111,275

$\bar{y}$ = 112.6

[a]Ref. Table XII; $\bar{y}_i = S_i/k_i$

[b]Ref. Figure 5(c)